Verification of photonic families of non-Gaussian entangled states

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When we consider entanglement verification beyond Gaussian states we may specify a set of non-Gaussian entangled states which cannot be verified to be inseparable by the measurement of the covariance matrix. This will specify the set of purely non-Gaussian entanglement whose quantum correlation cannot be explained by the correlation on the first and/or second order moments of canonical variables. Here we determine such a set of non-Gaussian entanglement for two classes of photonic two-mode states generated by using two of familiar Gaussian operations on the products of number states. We also find a couple of higher order entanglement criteria useful to detect such non-Gaussian entanglement.

I. INTRODUCTION

Quantum correlation or quantum entanglement plays a central role in quantum physics and quantum information science. A basic problem on quantum entanglement is the separability problem in which one is asked to determine whether a given quantum state is entangled or not. For quantum continuous-variable systems or bosonic modes a major goal of the separability problem was to determine the inseparability of Gaussian states [1]. It has turned out that the measurement of the covariance matrix (CM) is sufficient to detect Gaussian entanglement of a two-mode system [2, 3]. A bit interestingly, the Gaussian entanglement can be verified by considering Einstein-Podolsky-Rosen-Like (EPR-like) correlation [2, 4].

Beyond the measurement of the CM there have been many proposed entanglement criteria based on the measurements of higher order moments of canonical quadrature variables [5–10]. Generally, entanglement criteria with higher order moments are thought to be powerful to detect non-Gaussian entanglement, however, the entanglement criteria based on the CM work whether or not the state is Gaussian. Thus, it is not clear in what case of non-Gaussian states we need higher order moments for entanglement verification. Hence, it would be valuable to investigate limitations of entanglement criteria with the lower order moments of the CM and show advantages of higher order criteria [11–14].

In this report we review the results of Ref. [14] and some points of Ref. [15]. We consider two classes of non-Gaussian entangled states generated by the action of the beamsplitter or the two-mode squeezer on the product of number states. It is shown that, for many of these states, the CM is compatible with the CM of separable Gaussian states and their inseparability cannot be verified by the measurements of the first and second moments of canonical variables. We identify a couple of continuous-variable entanglement criteria with higher order moments [10] to verify these non-Gaussian entanglement.

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II. ENTANGLEMENT OF TWO-MODE GAUSSIAN STATES

We consider two mode bosonic fields described by a set of canonical variables $[\hat{x}_A, \hat{p}_A] = [\hat{x}_B, \hat{p}_B] = i$. We may use the bosonic field operators $[\hat{a}, \hat{a}^\dagger] = [\hat{b}, \hat{b}^\dagger] = 1$ related to the canonical variables by $\hat{a} = (\hat{x}_A + i\hat{p}_A)/\sqrt{2}$ and $\hat{b} = (\hat{x}_B + i\hat{p}_B)/\sqrt{2}$. We know that Gaussian states are basically important class of quantum states to demonstrate the property of quantum uncertainties $\langle \Delta^2 \hat{x} \rangle \langle \Delta^2 \hat{p} \rangle \geq 1/4$. They are completely characterized by the first and second moments of the canonical variables, and basic properties can be described by the covariance matrix (CM). The CM of a two-mode state ρ is a symmetric 4-by-4 matrix defined by

$$\gamma_\rho := \langle \Delta \hat{d} \Delta \hat{d}^t + (\Delta \hat{d} \Delta \hat{d}^t)^t \rangle_\rho =: \begin{pmatrix} A & C \\ C^t & B \end{pmatrix} \quad (1)$$

where $\hat{d} := (\hat{x}_A, \hat{p}_A, \hat{x}_B, \hat{p}_B)^t$. In terms of the CM the physical requirement due to the canonical commutation relations can be written as $\gamma + i\Omega \geq 0$ where

$$\Omega := \begin{pmatrix} J & 0 \\ 0 & J \end{pmatrix}, \quad J := \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}. \quad (2)$$

In the case of two-mode Gaussian states we can determine the entanglement of formation by the CM [16]. We can also determine whether a given two-mode Gaussian state is entangled or not by using the following Simon's separable condition [3]:

$$D := \det \gamma + 1 - \det A - \det B + 2 \det C \geq 0. \quad (3)$$

This is the necessary and sufficient condition for separability of two-mode Gaussian states. For non-Gaussian states, this is a necessary condition for separability, or equivalently, violation of Eq. (3) is sufficient to verify the existence of entanglement.

III. HIGHER ORDER SEPARABLE CONDITIONS

While Simon's criterion is useful to analyze entanglement of Gaussian states there are a number of entanglement criteria using the third or higher order moments of canonical variables. In this regard a remarkable result has been presented by Shchukin and Vogel [5]. Let us define the expectation-value matrix for the products of field operators:

$$\mathcal{M} = \begin{pmatrix} 1 & \langle a\rangle & \langle a^\dagger\rangle & \langle b^\dagger\rangle & \langle b\rangle & \dots \\ \langle a^\dagger\rangle & \langle a^\dagger a\rangle & \langle a^{\dagger 2}\rangle & \langle a^\dagger b^\dagger\rangle & \langle a^\dagger b\rangle & \dots \\ \langle a\rangle & \langle a^2\rangle & \langle a a^\dagger\rangle & \langle a b^\dagger\rangle & \langle ab\rangle & \dots \\ \langle b\rangle & \langle ab\rangle & \langle a^\dagger b\rangle & \langle b^\dagger b\rangle & \langle b^2\rangle & \dots \\ \langle b^\dagger\rangle & \langle ab^\dagger\rangle & \langle a^\dagger b^\dagger\rangle & \langle b^{\dagger 2}\rangle & \langle bb^\dagger\rangle & \dots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}.$$

According to Ref. [5], positivity of the determinant of any principal sub-matrix of $\mathcal{M}$ gives a separable condition. For example, in Ref. [13] a principle sub-matrix is chosen to give a separable condition of the form:

$$\begin{vmatrix} 1 & \langle ab^\dagger\rangle \\ \langle a^\dagger b\rangle & \langle a^\dagger a b^\dagger b\rangle \end{vmatrix} = \langle a^\dagger a b^\dagger b\rangle - \langle ab^\dagger\rangle\langle a^\dagger b\rangle \geq 0. \quad (4)$$

This condition also appeared in Eq. (8) of Ref. [7] (See also Sec. IIIC of Ref. [17]). In this manner we can generate a huge number of separable conditions by choosing principle minors from $\mathcal{M}$.

Now we are thought to be in the position to discuss the role of higher order moments for entanglement detection. Due to Simon's condition the measurement of the CM is sufficient for entanglement verification of the Gaussian states. One may say that we need the measurement of higher order moments for verification of non-Gaussian entanglement. However, Simon's condition is a necessary condition for separability and violation of it can signify the existence of entanglement for non-Gaussian states. Thus, we know little about the case where measurements of higher order moments are essential for entanglement detection. To find a specific role of higher order criteria we may consider the following two steps (See, Table I): Firstly, we find a set of the entangled states whose CMs are incompatible with CM of entangled Gaussian states. Secondly, we specify the set of higher order entanglement criteria for these non-Gaussian states. If the two steps are achieved we can say an advantage of higher order criteria for detection of non-Gaussian entanglement. In this research direction we can find a couple of results on Refs. [11–14].

Here, we employ the concept of the orthogonality/non-orthogonality to obtain a set of non-Gaussian entangled states [15]. Suppose that there is a set of Gaussian states. Then any pair of the elements are non-orthogonal to each other since Gaussian wave functions have non-zero overlaps. Conversely, if there is a pair of orthogonal states at least one of them is non-Gaussian states. Hence, if we have a set of orthogonal states, at most, one element is Gaussian and the other elements are non-Gaussian. On the basis of this fact, if we have an orthonormal basis consist of entangled states, at most, one element is Gaussian and we obtain a sequence of non-Gaussian entangled states. Note that, if one find such an entangled orthonormal basis, another many entangled orthonormal bases can be obtained by using non-Gaussian local unitary operations [15].

Entanglement	Measurement/Verification
Gaussian	1st&2nd order moments (CM)
Non-Gaussian	1st&2nd order moments (CM)
	higher order moments

TABLE I. Some of non-Gaussian entangled states can be verified to be entangled by the measurement of the covariance matrix due to the violation of Simon's condition, i.e, $D < 0$. Therefore, to find the role of higher order entanglement criterion, firstly it might be necessary to find a set of non-Gaussian entangled states which cannot violate Simon's condition. Secondary, it is an interesting task to search a set of higher order entanglement criteria which can verify such non-Gaussian entanglement.

IV. ENTANGLED ORTHONORMAL BASIS

The notion of the entangled orthonormal basis might be a natural element in quantum physics. A two-mode orthonormal basis can be made by a product of orthonormal bases of single modes. If $\{|n\rangle\}_n$ is an orthonormal basis $|\langle n|n'\rangle| = \delta_{n,n'}$ then

$$|n,m\rangle = |n\rangle \otimes |m\rangle \quad (5)$$

forms an orthonormal basis on the composite system as we have $|\langle n,m|n',m'\rangle| = \delta_{n,n'}\delta_{m,m'}$. When we apply a proper interaction unitary U on the product states we are likely to obtain an entangled orthonormal basis $\{U|n,m\rangle\}_{n,m}$ because correlated pure states are always entangled.

As a tractable example, we may consider products of number states and Gaussian unitary operation. For the product vacuum state $|0,0\rangle$, application of the Gaussian unitary U gives a Gaussian ground state, say $|g\rangle = U|0,0\rangle$. The excited states with non-zero photon $|e\rangle = U|n,m\rangle$ are orthogonal to the ground state,

$$\langle g|e\rangle = \langle 0,0| U^\dagger U |n,m\rangle = \langle 0,0|n,m\rangle = 0. \quad (6)$$

Hence, the excited states have non-Gaussian wave functions. For these states calculation of the canonical moments or an expectation value of other operators can be routinely done by repeatedly using the basic relations:

$$a|n\rangle = \sqrt{n}\,|n-1\rangle$$
$$a^\dagger|n\rangle = \sqrt{n+1}\,|n+1\rangle\,. \quad (7)$$

In the following, we consider two specific Gaussian unitary interactions illustrated in FIG. 1. One is the two-mode squeezing operation

$$V_\xi := e^{\xi a^\dagger b^\dagger - \xi^* ab}. \quad (8)$$

The other one is the beam splitter (BS) operation

$$U_r = e^{ra^\dagger b - r^* a b^\dagger}. \tag{9}$$

These unitary operators introduce a symmetric pair of orthonormal bases as in FIG. 2. Note that the interaction parameters of the unitary operators, transforms as

$$\begin{aligned} \xi &\to \xi e^{-i(\phi+\varphi)} \\ r &\to r e^{-i(\phi-\varphi)} \end{aligned} \tag{10}$$

under the rotation of local field operators

$$\begin{aligned} a &\to a e^{i\phi} \\ b &\to b e^{i\varphi}. \end{aligned} \tag{11}$$

Hence, we can choose the phases of the interaction parameters provided that the local Gaussian unitary operators can be optimized.

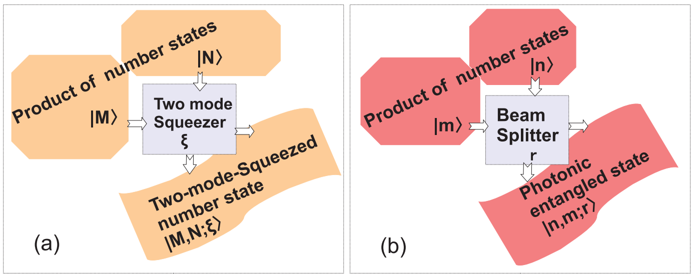

FIG. 1. Two families of photonic non-Gaussian entangled states are generated by the two-mode squeezing operation (a) or the beam-splitter operation (b) on the product of number states. They are respectively parameterized by the interaction parameter ξ or r.

A. Two-mode squeezed number states

The states $|M,N;\xi\rangle$ generated by the application of two-mode squeezer unitary $V_\xi = e^{\xi a^\dagger b^\dagger - \xi^* ab}$ on the product of number states $|M,N\rangle$ are called two-mode squeezed (TMS) number states. By introducing the coupled field operators

$$\begin{aligned} \hat{A}_\xi &:= V_\xi a V_\xi^\dagger = \frac{a - \xi b^\dagger}{\sqrt{1-|\xi|^2}}, \\ \hat{B}_\xi &:= V_\xi b V_\xi^\dagger = \frac{b - \xi a^\dagger}{\sqrt{1-|\xi|^2}}, \end{aligned} \tag{12}$$

we can write

$$|M,N;\xi\rangle = \frac{(\hat{A}_\xi^\dagger)^M}{\sqrt{M!}} \frac{(\hat{B}_\xi^\dagger)^N}{\sqrt{N!}} |\psi_\xi\rangle \tag{13}$$

where

$$|\psi_\xi\rangle_{AB} := V_\xi |0,0\rangle = \sqrt{1-|\xi|^2} \sum_{n=0}^{\infty} \xi^n |n\rangle_A |n\rangle_B, \tag{14}$$

is the well-known TMS vacuum state. The TMS number states are entangled whenever the interaction parameter ξ is non-zero [15]. From the construction, we have the orthonormal relation $\langle M', N'; \xi | M, N; \xi\rangle = \delta_{N,N'}\delta_{M,M'}$ and the wave function of $|M,N;\xi\rangle$ is non-Gaussian except for $M = N = 0$.

Since the two-mode squeezing operation can be expanded in the terms of the pair of annihilation operators ab and the pair of creation operators $a^\dagger b^\dagger$ it preserves relative photon number of the two local modes $N_a := a^\dagger a$ and $N_b := b^\dagger b$. This implies that a TMS number state can be written in coherent superposition of the product of the number basis with a fixed relative photon number as

$$|M,N;\xi\rangle = \sum_{m=-\min\{M,N\}}^{\infty} C_m |M+m\rangle_A |N+m\rangle_B. \tag{15}$$

Therefore, the TMS number state $|M,N;\xi\rangle$ is an eigenstate of the relative photon number $N_a - N_b$ belong to the eigenvalue of $M - N$. An explicit form of the Schmidt coefficients of the TMS number state $\{C_m\}$ can be found in Refs. [15, 18]. We note that the Schmidt rank of the TMS number states is infinite.

It is a simple exercise to determine the CM of the TMS number state. The four determinants of $|M,N;\xi\rangle$ are calculated to be

$$\begin{aligned} \det\gamma &= (1+2N^2)(1+2M^2), \\ \det A &= \frac{(1+2M+(1+2N)|\xi|^2)^2}{(1-|\xi|^2)^2}, \\ \det B &= \frac{(1+2N+(1+2M)|\xi|^2)^2}{(1-|\xi|^2)^2}, \\ \det C &= -\frac{4(1+M+N)^2|\xi|^2}{(1-|\xi|^2)^2}. \end{aligned} \tag{16}$$

This implies the violation of Simon's separable condition when

$$\left(M - \frac{|\xi|^2}{1-|\xi|^2}\right)\left(N - \frac{|\xi|^2}{1-|\xi|^2}\right) < \frac{|\xi|^2}{(1-|\xi|^2)^2}. \tag{17}$$

This inequality can be associated with an inverse proportional curve as shown in FIG. 3. Above this curve the TMS number states have the CMs being compatible with the CMs of separable Gaussian states. Hence, we can see that a large part of the TMS number states cannot be determined to be entangled by Simon's condition. This shows a limitation of the measurement of the CM and a potential advantage of measuring higher order moments for entanglement detection.

Role of Duan's sum condition

It might be instructive to test the sum criterion based on the EPR-like uncertainties given by Duan *et al.* [2]

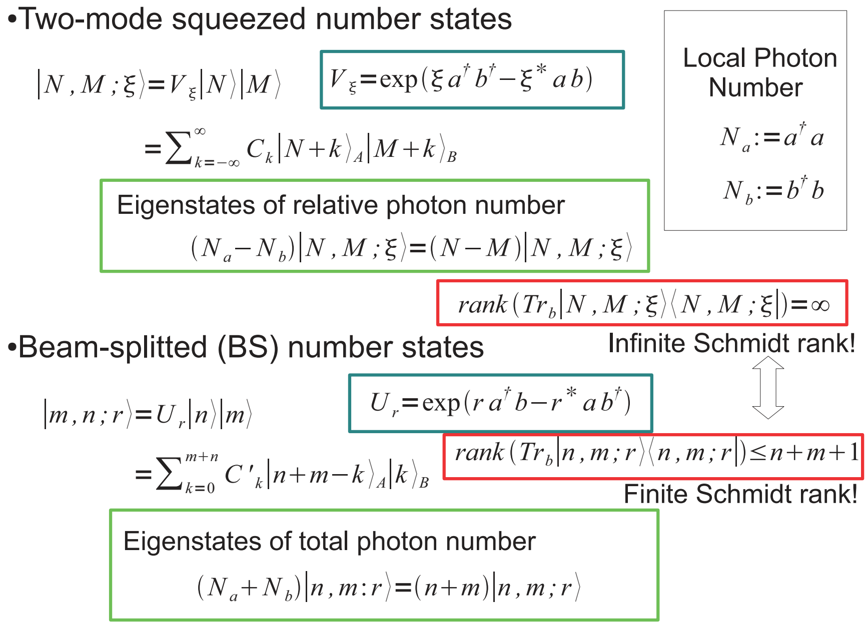

FIG. 2. Dual properties of the two orthonormal bases.

(See also, [4, 12, 15]). In our notation, the sum criterion for separable states can be written as [15]

$$\left\{2\left(\langle \hat{A}_\zeta^\dagger \hat{A}_\zeta\rangle - \langle \hat{A}_\zeta^\dagger\rangle\langle \hat{A}_\zeta\rangle\right)+1\right\} \ge \frac{1+\zeta^2}{1-\zeta^2}. \quad (18)$$

Until the end of this section we assume $0<\zeta,\xi<1$ for simplicity.

Using the following relation

$$\hat{A}_\zeta^\dagger \hat{A}_\zeta = a^\dagger a + \zeta^2 b^\dagger b - \zeta(a^\dagger b^\dagger + ab)/(1-\zeta^2)$$
$$= \frac{1}{1-\zeta^2}\frac{1}{1-\xi^2}\{(1-\zeta\xi)^2 \hat{A}_\xi^\dagger \hat{A}_\xi + (\xi-\zeta)^2 \times (\hat{B}_\xi^\dagger \hat{B}_\xi + 1) + (1-\zeta\xi)(\xi-\zeta)(\hat{A}_\xi \hat{B}_\xi + \hat{A}_\xi^\dagger \hat{B}_\xi^\dagger)\}$$

we have the mean value of $\hat{A}_\zeta^\dagger \hat{A}_\zeta$ for the TMS number state $|M,N;\xi\rangle$ as

$$\langle \hat{A}_\zeta^\dagger \hat{A}_\zeta\rangle = \frac{1}{1-\zeta^2}\frac{1}{1-\xi^2}\left\{(1-\zeta\xi)^2 M + (\xi-\zeta)^2 N\right\}.$$

From this relation the separable condition of Eq. (18) for the TMS number state $|M,N;\xi\rangle$ is shown to be violated when

$$(1-\zeta\xi)^2 M + (\xi-\zeta)^2 N < \zeta^2(1-\xi^2). \quad (19)$$

This condition gives a line tangent to the boundary of Eq. (17). The regime determined by Eq. (17) can be also determined when we vary the parameter ζ of Eq. (18) as shown in FIG. 3.

To demonstrate this, let us consider two tangent lines

$$(1-\zeta\xi)^2 M + (\xi-\zeta)^2 N = \zeta^2(1-\xi^2)$$
$$(1-\zeta'\xi)^2 M + (\xi-\zeta')^2 N = {\zeta'}^2(1-\xi^2). \quad (20)$$

The intersection of this two lines specifies a point on the boundary in the limit of $\zeta'\to\zeta$. It is given by

$$M = \frac{\zeta\xi}{1-\zeta\xi}$$
$$N = \frac{\xi}{\zeta-\xi}. \quad (21)$$

By eliminating the parameter ζ from this expression we obtain the boundary curve

$$\left(M-\frac{|\xi|^2}{1-|\xi|^2}\right)\left(N-\frac{|\xi|^2}{1-|\xi|^2}\right) = \frac{|\xi|^2}{(1-|\xi|^2)^2}. \quad (22)$$

Hence, the detectable set of the TMS number states for Duan's criterion is the same as the set for Simon's criterion. In this regard, we observe no essential difference in entanglement detection of the TMS number states between Simon's criterion and Duan's criterion.

B. Beam-splitted number states

Similar to the case of TMS number states we may define beamsplitted (BS) number states by application of U_r of Eq. (9) on the product of number states as [14]

$$|n,m;r\rangle := U_r|n,m\rangle = \frac{(c_r^\dagger)^n}{\sqrt{n!}}\frac{(d_r^\dagger)^m}{\sqrt{m!}}|0,0\rangle, \quad (23)$$

where

$$c_r := U_r a U_r^\dagger = \frac{a-rb}{\sqrt{1+|r|^2}},\ d_r := U_r b U_r^\dagger = \frac{r^* a + b}{\sqrt{1+|r|^2}}. \quad (24)$$

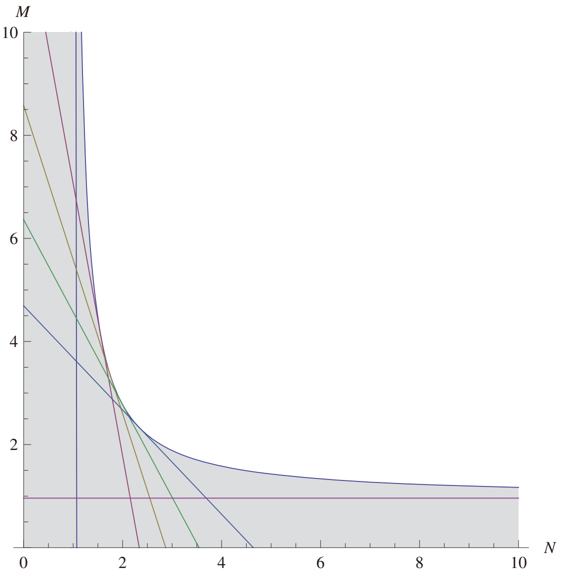

FIG. 3. A pair of non-negative integers M and N can be assigned for a TMS number state $|M, N; \xi\rangle$ of Eq. (13). Below the inverse proportional curve, the TMS number states have the covariance matrices being compatible to the covariance matrices of the entangled Gaussian states, and can be confirmed to be entangled from the measurements of the covariance matrix. Here, we displayed the case of $\xi = 0.7$ in Eq. (17). The tangent lines are due to the sum condition of Eq. (19) with $\zeta = 0.72, 0.87, 0.91, 0.95$, and 0.999. A line nearly parallel to the vertical N-axis can be obtained by replacing the role of M and N in Eq. (19).

Its inseparability can be proven from the theorem for beam-splitter entangler [19] (see also [20]). We assume that the interaction is non zero $r \neq 0$ or different from the complete swapping operation $r \neq \infty$. Note that, in the case of $n = m = 0$, the BS number state is the product of vacuum states and separable. Since the beamsplitter operation U_r can be expanded in the terms of the annihilation and creation pairs of each mode, $ab^\dagger$ and $a^\dagger b$, it preserves the total photon number $N_a + N_b$. This implies that a BS number state can be written in the superposition of the number-state products with a fixed total photon number as

$$|m, n; r\rangle = \sum_{k=0}^{m+n} C'_m |n + m - k\rangle_A |k\rangle_B. \tag{25}$$

It tell us that the BS number state is an eigenstate of the total photon number $N_a + N_b$ belong to the eigenvalue of $m + n$. The number of non-zero Schmidt coefficients for $|m, n; \xi\rangle$ is at most $m + n + 1$, and the Schmidt rank of the BS number state is finite in contrast to TMS number states. Such a dual property on TMS number states and BS number states is summarized in FIG. 2.

It is a simple exercise to determine the CM γ of Eq. (1) for the BS number state [14]. We can obtain the expression of the four determinants

$$\begin{aligned} \det \gamma &= (1 + 2n^2)(1 + 2m^2), \\ \det A &= \frac{(1 + 2n + (1 + 2m)|r|^2)^2}{(1 + |r|^2)^2}, \\ \det B &= \frac{(1 + 2m + (1 + 2n)|r|^2)^2}{(1 + |r|^2)^2}, \\ \det C &= \frac{4(m - n)^2 |r|^2}{(1 + |r|^2)^2}. \end{aligned} \tag{26}$$

This implies that there is no violation of Simon's condition Eq. (3) for the BS number states although they are thought to be full of entanglement.

Note that two-mode Gaussian entanglement can be associated with the existence of a stronger EPR-like correlation as described in Ref. [4]. In this regard, the insensitivity to Simon's criterion suggests that the states exhibit no EPR-like correlation. Then, it is natural to ask what types of correlation or state properties are relevant to such non-Gaussian entanglement. We will address this point in next section. In the rest of this section we introduce a simple higher order criterion which is sensitive to a set of the BS number states.

An advantage of the condition by Hillery and Zubairy [7]

It might be interesting that a relation similar to Eq. (17) can be derived for the BS number states when we use the separable condition of Eq. (4). For the BS number states of Eq. (23) we have

$$\begin{aligned} \langle a^\dagger a b^\dagger b \rangle &= \left(\frac{1}{1 + |r|^2} \right)^2 \Big\{ (1 - |r|^2)^2 mn \\ &\quad + |r|^2 [m(m-1) + n(n-1)] \Big\}, \\ \langle a b^\dagger \rangle &= \frac{r}{1 + |r|^2} (m - n). \end{aligned} \tag{27}$$

Then, the violation of the separable condition of Eq. (4) turns out to be

$$\left(m - \frac{|r|^2}{1 + |r|^4} \right) \left(n - \frac{|r|^2}{1 + |r|^4} \right) < \left(\frac{|r|^2}{1 + |r|^4} \right)^2 . \tag{28}$$

This inequality is quite similar to the inequality of Eq. (17). The condition of Eq. (28) can verify entanglement for the cases of $(m, n) = \{(k, 0), (0, k)\}$ with $k = 1, 2, 3, \cdots$. This is an advantage of the measurements of higher order moments for detection of purely non-Gaussian entanglement although it covers a small portion of the BS number states.

V. VERIFICATION OF NON-GAUSSIAN ENTANGLEMENT

In the previous sections we have tested Simon's condition for two set of non-Gaussian states and identified the entangled states whose CMs are compatible with separable Gaussian states. Now we will identify a set of higher order criteria that can verify the inseparability of these non-Gaussian entangled states.

We consider two separable conditions related to the generators of SU(2) and SU(1,1) algebra [6–10, 14, 21, 22],

$$\left[\langle \Delta^2(J_y)\rangle + \frac{1}{4}\right]\langle \Delta^2 K_z\rangle \geq \frac{1}{4}|\langle J_x\rangle|^2, \qquad (29)$$

$$\left[\langle \Delta^2(K_y)\rangle - \frac{1}{4}\right]\langle \Delta^2 J_z\rangle \geq \frac{1}{4}|\langle K_x\rangle|^2, \qquad (30)$$

where the generators of SU(2) and SU(1,1) algebra are defined as

$$\begin{aligned} J_x &= \frac{1}{2}(a^\dagger b + ab^\dagger), \quad K_x = \frac{1}{2}(a^\dagger b^\dagger + ab), \\ J_y &= \frac{1}{2i}(a^\dagger b - ab^\dagger), \quad K_y = \frac{1}{2i}(a^\dagger b^\dagger - ab), \\ J_z &= \frac{1}{2}(N_a - N_b), \quad K_z = \frac{1}{2}(N_a + N_b + 1). \qquad (31) \end{aligned}$$

In the final line we use the notation for local photon-number operators $N_a = a^\dagger a$ and $N_b = b^\dagger b$. Note that J_y and K_y are thought to be the Hamiltonians of a beam-splitter and a two-mode Squeezer, respectively. The separable conditions Eqs. (29) and (30) can be derived by taking partial transposition on the spin-like uncertainty relations $\langle \Delta^2 K_y\rangle\langle \Delta^2 K_z\rangle \geq \frac{1}{4}|\langle K_x\rangle|^2$ and $\langle \Delta^2 J_y\rangle\langle \Delta^2 J_z\rangle \geq \frac{1}{4}|\langle J_x\rangle|^2$ (See, e.g., [10, 14]). We next show that TMS number states violate the first inequality of Eq. (29) and that a large part of BS number states violates the second inequality of Eq. (30).

The TMS number states are eigenstates of the relative photon number $N_a - N_b$ and so are the eigenstates of J_z. This implies that uncertainty of J_z is zero and the term in the LHS of Eq. (29) vanishes for the TMS number states. When we calculate the term in the RHS of Eq. (29) for the TMS number states, we can check that it is positive for any N and M. Consequently, this higher order criterion can verify the inseparability of any TMS number state. This shows a clear advantage of the higher order criterion compared with the criterion based on the CM.

Similar relation can be found between the BS number states and the inequality of Eq. (30). The BS number states are eigenstates of the total photon number $N_a + N_b$ and so are the eigenstates of K_x. This implies the uncertainty of K_x is zero and the LHS of Eq. (30) vanishes for the BS number states. When we calculate the term in the RHS of Eq. (30) for the BS number states, we can check that it is positive whenever $n \neq m$. Consequently, this higher order criterion verifies the inseparability of the BS number states except for the small potion of $n = m$. Before to discuss the case of $m = n$ in detail we would like to mention the mechanism of the present entanglement detection.

The mechanism that the non-Gaussian entangled states violate the higher order separable conditions can be summarized as follows (See FIG. 4). On one hand, the terms $\langle \Delta^2 K_z\rangle$ and $\langle \Delta^2 J_z\rangle$ in the LHSs of Eqs. (29) and (30) correspond to the fluctuations of sum or difference of the local photon number operators. Thereby, they become smaller as the classical correlations on the photon numbers become stronger. On the other hand, the terms $\langle J_x\rangle$ and $\langle K_x\rangle$ in the RHSs of Eqs. (29) and (30) show coherence between the product of number states. More precisely, the J_x operator contributes to the coherent exchange of a single photon between the local modes due to the operators $ab^\dagger$ and $a^\dagger b$; The K_x operator contributes to the coherent pair creation or pair annihilation of the photons due to the operators $a^\dagger b^\dagger$ and ab. Such coherent superpositions on the number basis can be generated by the beamsplitter unitary and the two-mode squeezing unitary, respectively. In the present case, the states are keeping the coherence and the classical correlation simultaneously, and this is the key mechanism of the entanglement verification. Interestingly, it still shows a simple physics beyond the Gaussian paradigm. The new mechanism covers a part of entangled states which have no EPR correlation, and it might be an interesting task to identify whole class of non-Gaussian entanglement relevant to this type of entanglement verification. On the other hand, it might be also interesting to identify to what extent this mechanism is useful to detect Gaussian entanglement.

Now we go back to the case of the BS number states with $n = m$. We can show that they violate another higher order separable condition proposed in Ref. [10] (See also [4]):

$$\left[\langle \Delta^2(\tilde{L}_y)\rangle + \langle N_{22}\rangle\right]\langle \Delta^2 N_+\rangle \geq \frac{1}{4}|\langle \tilde{L}_x\rangle|^2, \qquad (32)$$

where

$$\begin{aligned} N_{22} &= \frac{1}{4}[a^2, (a^\dagger)^2] \otimes [b^2, (b^\dagger)^2], \quad N_+ = \frac{1}{4}(N_a + N_b), \\ \tilde{L}_x &= \frac{1}{2}[(a^\dagger b)^2 + (ab^\dagger)^2], \quad \tilde{L}_y = \frac{1}{2i}[(a^\dagger b)^2 - (ab^\dagger)^2]. \end{aligned} \qquad (33)$$

Since N_+ is proportional to the total photon number operator its fluctuation is zero for BS number states. Similar to the case of Eq. (30), the LHS of Eq. (32) vanishes for BS number states. For the RHS term of Eq. (32) we have

$$\langle \tilde{L}_x\rangle = \frac{r^2 + (r^*)^2}{2(1+|r|^2)^2}[m(m-1) + n(n-1) - 4nm]. \qquad (34)$$

When we set $n = m$ it implies

$$|\langle \tilde{L}_x\rangle| = \frac{|r^2 + (r^*)^2|}{(1+|r|^2)^2} n(n+1). \qquad (35)$$

This concludes that the BS number states with $n = m > 0$ violate the separable condition Eq. (32). The mechanism of the violation of Eq. (32) can also be associated

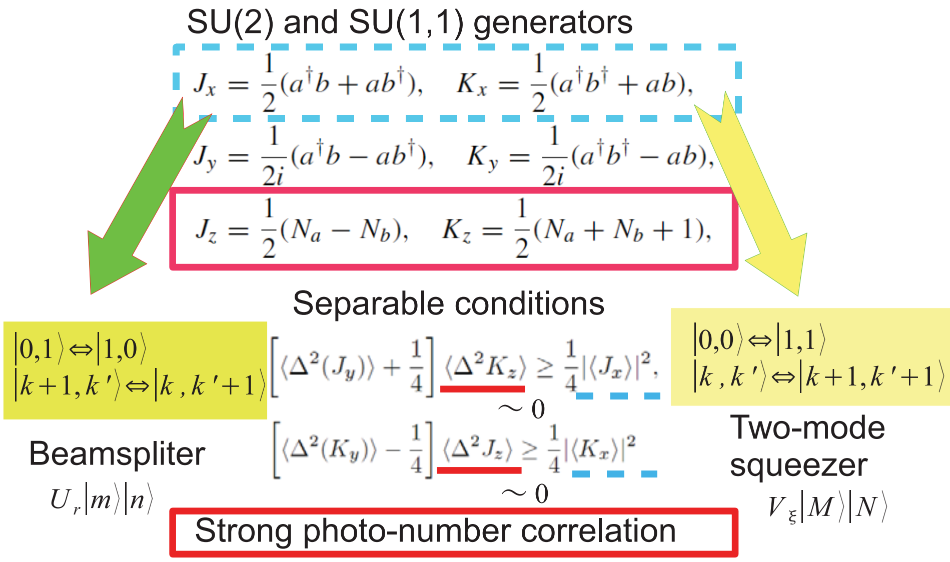

FIG. 4. Smaller fluctuations $\Delta J_z \sim 0$ and $\Delta K_z \sim 0$ imply stronger photon number correlation whereas non-zero contributions of J_x and K_x imply coherence between the number basis of the same total photon number or the same difference photon number, respectively. The coexistence of strong classical correlation and the coherence induces the violation of the separable conditions and signifies the existence of entanglement.

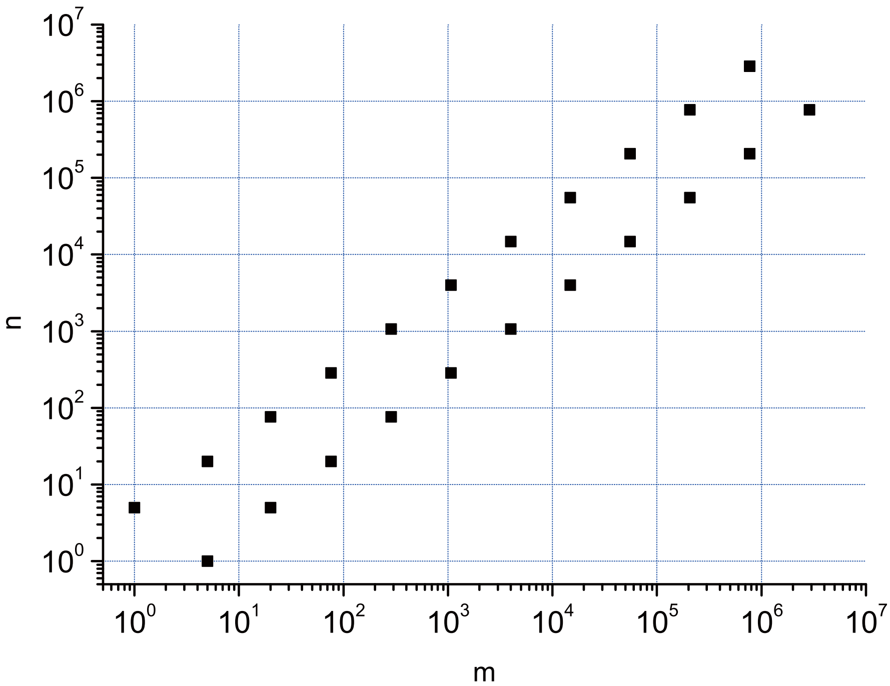

FIG. 5. Distribution of (m, n) that satisfies Eq. (36). There are 12 solutions for $0 \leq m < n \leq 10^7$: (0,1), (1,5), (5, 20), (20, 76), (76, 285), (285, 1065), (1065, 3976), (3976, 14840), (4840, 55385), (55385, 206701), (206701, 771420), and (771420, 2878980). The solutions $(m, n) = \{(0, 1), (1, 0)\}$ are not displayed in this figure.

with the coexistence of the coherence and the classical correlation. In this case the $\tilde{L}_x$ operator can be related to the coherence between the processes of the two photon exchange due to the terms $(ab^\dagger)^2$ and $(a^\dagger b)^2$.

Note that there is a very small portion of the BS number states which cannot violate the separable condition Eq. (32). Those states can be specified by $\langle \tilde{L}_x \rangle = 0$ in Eq. (34), namely, they are determined by

$$m(m-1) + n(n-1) - 4nm = 0. \tag{36}$$

This equation is independent of the interaction parameter r, and the solutions are thought to specify a general property on the BS number states. A set of solutions of this equation is shown in FIG. 5. In the Logarithm scale, they seem to be distributed with a regular interval, and the number of states are fairly said to be very small.

VI. SUMMARY

We have introduced the notion of entangled orthonormal basis to discuss the role of higher order moments of canonical operators for entanglement detection. We have considered two examples of entangled orthonormal bases which can be generated by using Gaussian interactions for products of number states. We have tested Simon's criteria and shown limitations of entanglement verification scheme based on the covariance matrix. We have also found a couple of higher order separable conditions which have advantage on verification of those non-Gaussian entanglement. The mechanism of entanglement detection are addressed to be the coexistence of the coherence and classical correlation. This mechanism is different from the mechanism of the entanglement detection for Gaussian states where an EPR-like correlation plays a key role. It is an open question to further identify the interrelation between the mechanisms.

Finally, the following points could be emphasized: We can go beyond the Gaussian paradigm of entanglement detection by a combination of basic elements of quantum optics, such as beamsplitter, two-mode squeezer, and number states. Interestingly, we can still enjoy a simple physics of entanglement detection over there. Experimental generation of number states and implementation of Gaussian unitary gates are essential technology in quantum optics and photonic quantum information science. Hence, experimental generation and observation of the states belong to the entangled orthonormal bases are thought to form an attractive field to demonstrate the feasibility of the experimental methods.

ACKNOWLEDGMENTS

This work was supported by the Grant-in-Aid for the Global COE Program "The Next Generation of Physics, Spun from Universality and Emergence" from the Ministry of Education, Culture, Sports, Science and Technology (MEXT) of Japan.

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