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***Abstract*—Recently a numerical calculation method for finding the minimax solution to the minimax problem in quantum signal detection was reported [10]. In this paper, we evaluate the error performance of the minimax receiver for 16QAM coherent state signal by using this calculation method. Through this numerical simulation, it will be pointed out that the use of the minimax strategy has an advantage rather than that of the Bayes strategy in designing quantum communication systems.**

I. Introduction

Quadrature amplitude modulation (QAM) format is widely used in current digital tele-communication systems [1]. As the coherent detection technology for optical communication systems is developed, it has also become one of important modulation formats in the field of optical communications [2]. In quantum communication theory the first theoretical analysis of QAM coherent state signal was done by Tamagawa University group including the author with the square-root measurement technique [3]. After this work, the Holevo capacity of 16QAM coherent state signal was numerically computed [4]. In 2010, the error performance of the square-root measurement for QAM coherent state signal in the presence of thermal noise was investigated by Cariolaro and Pierobon [5]. Moreover, QAM-based quantum stream cipher scheme was proposed in 2005 [6]. Thus, QAM is one of attractive modulation formats also in quantum communication theory.

In this paper, we focus on the error performance evaluation of QAM coherent state signal. In the literature [3], we have used the square-root measurement as a receiver. In general, the square-root measurement is not the optimal receiver that minimizes the average probability of decision errors for a given probability distribution of the signal, although it is considered as a sub-optimal receiver [7]. Therefore, we aim to evaluate the error performance of QAM coherent state signal in a sense of the 'optimal'. In many situations that have been done so far, the word 'optimal' means the use of the Bayes strategy in quantum signal detection theory. However, if one employs the Bayes strategy to design a quantum communication system, appropriate estimation of the probability distribution of the signals is needed in advance. Further, even if an appropriate distribution of the signals was given in advance, the receiver that is designed according to the Bayes strategy with this distribution will indicate the expected performance in only the case that the true distribution is identical to the distribution that was used in its design. To resolve this problem, we can employ the minimax strategy in quantum signal detection.

Necessary and sufficient conditions for the minimax solution to the minimax problem in quantum signal detection were first derived under the situation that the average probability of decision errors is used as its quality function [8]. Recently it was extended to more general case in which not only the average probability of decision errors but also the average Bayes-cost can be used as a quality function [9]. Moreover, a numerical calculation method for finding the minimax solution was reported [10]. In this study we attempt to evaluate the error performance of the minimax receiver for QAM coherent state signal by using this numerical calculation method. As the first step, we treat the case of 16QAM coherent state signal in this paper.

The remaining part of this paper is organized as follows. In Section II, we summarize the minimax problem in quantum signal detection and the necessary and sufficient conditions of the minimax solution in a general description. In Section III, we give a brief explanation of the numerical calculation method. The error performance of the minimax receiver for 16QAM coherent state signal is shown in Section IV, and finally we give a conclusion based on the simulation result in Section V.

II. The Minimax Problem in Quantum Signal Detection

Suppose that there are M hypotheses about states of a quantum system and the ith hypothesis H_i is the proposition that the system is in the state $\hat{\rho}_i$, where $\hat{\rho}_i$ is a density operator on Hilbert space $\mathcal{H}$: $\hat{\rho}_i \geq 0$ and $\mathrm{Tr}\hat{\rho}_i = 1$. The prior probability of hypothesis H_i is denoted by p_i. We let

$$\Pi = (\hat{\Pi}_1, \hat{\Pi}_2, \ldots, \hat{\Pi}_M) \tag{1}$$

be a positive operator-valued measure (POVM) consisting of M detection operators, where

$$\hat{\Pi}_j \geq 0 \quad \forall j, \quad \sum_{j=1}^{M} \hat{\Pi}_j = \hat{1}, \tag{2}$$

and $\hat{1}$ is the identity operator on $\mathcal{H}$. Then the conditional probability that the receiver chooses H_j when H_i is true is given by $P(j|i) = \mathrm{Tr}\hat{\rho}_i\hat{\Pi}_j$. The cost incurred by choosing

H_j when H_i is true is denoted by B_{ji}, which is a non-negative number. In general, $B_{ji} > B_{ii}$ if $i \neq j$. Letting $P(i,j) = p_i \mathrm{Tr}\hat{\Pi}_j \hat{\rho}_i$, the expected value of the Bayes costs is given by

$$\bar{B}(\Pi, \mathbf{p}) = \mathsf{E}[B_{ji}] = \sum_{i=1}^{M}\sum_{j=1}^{M} B_{ji} P(i,j), \tag{3}$$

where $\mathbf{p} = (p_1, p_2, \ldots, p_M)$. Defining the risk operators by

$$\hat{W}_i = \sum_{k=1}^{M} p_k B_{ik} \hat{\rho}_k, \quad 1 \leq i \leq M, \tag{4}$$

this average Bayes cost is rewritten as

$$\bar{B}(\Pi, \mathbf{p}) = \mathrm{Tr} \sum_{k=1}^{M} \hat{W}_k \hat{\Pi}_k. \tag{5}$$

If the Bayes cost is taken as $B_{ij} = 1 - \delta_{ij}$, the average Bayes cost becomes the average probability of decision errors $\bar{P}_{\mathrm{e}}(\Pi, \mathbf{p})$. The Bayes problem in quantum signal detection is expressed as follows:

$$\min_{\Pi \in \mathcal{D}} \bar{B}(\Pi, \mathbf{p}), \tag{6}$$

where we assume that $\mathbf{p}$ is given and where

$$\mathcal{D} = \left\{ \Pi = (\hat{\Pi}_1, \ldots, \hat{\Pi}_M) \right\} \tag{7}$$

is the set of all decision rules consisting of M decision operators. For this problem, it has been proved [11] that there exists an optimal decision rule Π^{bayes} such that

$$\min_{\Pi \in \mathcal{D}} \bar{B}(\Pi, \mathbf{p}) = \bar{B}(\Pi^{\mathrm{bayes}}, \mathbf{p}). \tag{8}$$

We call this decision rule Π^{bayes} the Bayes optimal receiver in this paper. Necessary and sufficient conditions for Π^{bayes} are given as follows [11] (See also [12], [15]):

$$\begin{aligned} \hat{\Pi}_i^{\mathrm{bayes}} (\hat{W}_i - \hat{W}_j) \hat{\Pi}_j^{\mathrm{bayes}} &= 0 \quad \forall (i,j), & (9)\\ \hat{W}_i - \hat{\Upsilon}^{\mathrm{bayes}} &\geq 0 \quad \forall i, & (10)\end{aligned}$$

where

$$\hat{\Upsilon}^{\mathrm{bayes}} = \sum_{k=1}^{M} \hat{W}_k \hat{\Pi}_k^{\mathrm{bayes}}. \tag{11}$$

At that time, the minimum average Bayes cost given $\mathbf{p}$ is

$$\bar{B}_{\min}(\mathbf{p}) = \min_{\Pi \in \mathcal{D}} \bar{B}(\Pi, \mathbf{p}) = \mathrm{Tr}\hat{\Upsilon}^{\mathrm{bayes}}. \tag{12}$$

Note that the set of the conditions (9) and (10) can be replaced to one of the equivalent sets of the conditions according to the literature [12]. Furthermore, $\bar{B}_{\min}(\mathbf{p})$ is a concave function of $\mathbf{p}$ over the convex set $\mathcal{P} = \{\mathbf{p} = (p_1, \ldots, p_M)\}$.

Now we move our attention to the minimax problem in quantum signal detection. We assume that the true probability distribution of the M quantum states is unknown. Under this assumption, we consider the following optimization problem.

$$\min_{\Pi \in \mathcal{D}} \max_{\mathbf{p} \in \mathcal{P}} \bar{B}(\Pi, \mathbf{p}). \tag{13}$$

Then we have the following results [9]:

Theorem 1: There exist Π° and $\mathbf{p}^\circ$ such that

$$\min_{\Pi \in \mathcal{D}} \max_{\mathbf{p} \in \mathcal{P}} \bar{B}(\Pi, \mathbf{p}) = \bar{B}^\circ = \max_{\mathbf{p} \in \mathcal{P}} \min_{\Pi \in \mathcal{D}} \bar{B}(\Pi, \mathbf{p}), \tag{14}$$

where $\bar{B}^\circ = \bar{B}(\Pi^\circ, \mathbf{p}^\circ)$. □

Hereafter, we call Π° the minimax decision rule, $\mathbf{p}^\circ$ the minimax distribution, and $\bar{B}^\circ$ the minimax value. Note that $\mathbf{p}^\circ$ is not necessary to be identical to the true distribution.

Theorem 2: Necessary and sufficient conditions for the minimax distribution and minimax receiver are

$$\begin{aligned} \hat{\Pi}_i^\circ (\hat{W}_i^\circ - \hat{W}_j^\circ) \hat{\Pi}_j^\circ &= 0 \quad \forall (i,j), & (15)\\ \hat{W}_i^\circ - \hat{\Upsilon}^\circ &\geq 0 \quad \forall i, & (16)\\ \sum_{j=1}^{M} B_{ji} \mathrm{Tr}\hat{\Pi}_j^\circ \hat{\rho}_i &= \mathrm{Tr}\hat{\Upsilon}^\circ \quad \forall i \text{ s.t. } p_i^\circ > 0, & (17)\\ \sum_{j=1}^{M} B_{ji} \mathrm{Tr}\hat{\Pi}_j^\circ \hat{\rho}_i &\leq \mathrm{Tr}\hat{\Upsilon}^\circ \quad \forall i \text{ s.t. } p_i^\circ = 0, & (18)\end{aligned}$$

where

$$\begin{aligned} \hat{W}_j^\circ &= \sum_{k=1}^{M} p_k^\circ B_{jk} \hat{\rho}_k \quad \forall j,\\ \hat{\Upsilon}^\circ &= \sum_{k=1}^{M} \hat{W}_k^\circ \hat{\Pi}_k^\circ. \end{aligned} \tag{19}$$

The minimax value is given by $\bar{B}^\circ = \mathrm{Tr}\hat{\Upsilon}^\circ$. □

The conditions (15) and (16) come from the fact that the minimax strategy is a special case of the Bayes strategy for $\mathbf{p}^\circ$. Therefore, this set of the conditions can also be replaced to an equivalent one.

Theorem 3: Let $\Pi^\circ = (\hat{\Pi}_1^\circ, \hat{\Pi}_2^\circ, \ldots, \hat{\Pi}_M^\circ)$ be the minimax receiver, and let $\mathbf{p}^\circ = (p_1^\circ, p_2^\circ, \ldots, p_M^\circ)$ be the corresponding minimax distribution. Then we obtain

$$\bar{B}_{\min}(\mathbf{q}) \leq \bar{B}(\Pi^\circ, \mathbf{q}) \leq \bar{B}^\circ \quad \forall \mathbf{q} \in \mathcal{P}. \tag{20}$$

If $p_i^\circ > 0$ for all i, then $\bar{B}(\Pi^\circ, \mathbf{q}) = \bar{B}^\circ$ for any $\mathbf{q}$ in $\mathcal{P}$. □

In the following sections, we restrict ourselves to the case of the average probability of decision errors; *i.e.* we let $B_{ij} = 1 - \delta_{ij}$. Hence, we use the error probability version of the set of the necessary and sufficient conditions instead of the set of Eqs. (15)-(18):

$$\begin{aligned} \hat{\Pi}_i^\circ (p_i^\circ \hat{\rho}_i - p_j^\circ \hat{\rho}_j) \hat{\Pi}_j^\circ &= 0 \quad \forall (i,j), & (21)\\ \mathrm{Tr}\hat{\Pi}_i^\circ \hat{\rho}_i &= \mathrm{Tr}\hat{\Gamma}^\circ \quad \forall i \text{ s.t. } p_i^\circ > 0, & (22)\\ \mathrm{Tr}\hat{\Pi}_i^\circ \hat{\rho}_i &\geq \mathrm{Tr}\hat{\Gamma}^\circ \quad \forall i \text{ s.t. } p_i^\circ = 0, & (23)\end{aligned}$$

where we have used the result of [12] (and [13]) to remove the condition (16), and where

$$\hat{\Gamma}^\circ = \sum_{k=1}^{M} p_k^\circ \hat{\rho}_k \hat{\Pi}_k^\circ. \tag{24}$$

At that time, the minimax value is given as $\bar{P}_{\mathrm{e}}^\circ = 1 - \mathrm{Tr}\hat{\Gamma}^\circ$.

III. A Numerical Calculation Method for Finding The minimax Solution

In this section we explain a numerical calculation method for finding the minimax solution $(\Pi^\circ, \mathbf{p}^\circ, \bar{P}_{\mathrm{e}}^\circ)$.

To begin with, we let $\mathbf{p}^{(0} = (p_1^{(0}, p_2^{(0}, \ldots, p_M^{(0)})$ be an arbitrarily chosen distribution of the signals. Then, we can find a Bayes decision rule $\Pi^{(0} = (\hat{\Pi}_1^{(0}, \hat{\Pi}_2^{(0}, \ldots, \hat{\Pi}_M^{(0)})$ for $\mathbf{p}^{(0}$; that is,

$$\min_{\Pi \in \mathcal{D}} \bar{P}_{\mathrm{e}}(\Pi, \mathbf{p}^{(0)}) = \bar{P}_{\mathrm{e}}(\Pi^{(0}, \mathbf{p}^{(0)}) = \bar{P}_{\mathrm{e}}^{(0}. \tag{25}$$

The calculation algorithm starts with these initial values $(\Pi^{(0}, \mathbf{p}^{(0)})$ and iterates the procedures mentioned below until the change of the average probability of decision errors is small enough.

Suppose that the calculation algorithm has reached to the $(n+1)$-th stage. At that time, the n-th distribution $\mathbf{p}^{(n} = (p_1^{(n}, p_2^{(n}, \ldots, p_M^{(n)})$ and the corresponding Bayes decision rule $\Pi^{(n} = (\hat{\Pi}_1^{(n}, \hat{\Pi}_2^{(n}, \ldots, \hat{\Pi}_M^{(n)})$ have been already recorded as the previously obtained data:

$$\min_{\Pi} \bar{P}_{\mathrm{e}}(\Pi, \mathbf{p}^{(n)}) = \bar{P}_{\mathrm{e}}(\Pi^{(n}, \mathbf{p}^{(n)}) = \bar{P}_{\mathrm{e}}^{(n}. \tag{26}$$

Now we choose a pair of indeces (i, j) such that $i \neq j$. For this (i, j), we consider the probability distributions of the form

$$\mathbf{p}^{(\mathrm{temp}} = (p_1^{(n}, p_2^{(n}, \ldots, \underbrace{x}_{i\mathrm{th}}, \ldots, \underbrace{y}_{j\mathrm{th}}, \ldots, p_M^{(n)}), \tag{27}$$

where x and y satisfy the conditions of $x \geq 0$, $y \geq 0$, and $x + y = 1 - \sum_{k \neq i, \neq j} p_k^{(n}$. From the concavity of $\bar{P}_{\mathrm{e}}^{\mathrm{bayes}}(\mathbf{p})$, we have the inequality

$$\begin{aligned} \bar{P}_{\mathrm{e}}(\Pi^{(n}, \mathbf{p}^{(n)}) &\leq \max_{x,y} \bar{P}_{\mathrm{e}}^{\mathrm{bayes}}(\mathbf{p}^{(\mathrm{temp})}) \\ &= \max_{x,y} \min_{\Pi} \bar{P}_{\mathrm{e}}(\Pi, \mathbf{p}^{(\mathrm{temp})}). \end{aligned} \tag{28}$$

Therefore, our task at this stage is to find a pair $(\Pi^*, \mathbf{p}^*)$ such that

$$\max_{x,y} \min_{\Pi} \bar{P}_{\mathrm{e}}(\Pi, \mathbf{p}^{(\mathrm{temp})}) = \bar{P}_{\mathrm{e}}(\Pi^*, \mathbf{p}^*). \tag{29}$$

Since x and y are not independent, the maximization part is a single-parameter maximization problem. That is, by changing the value of x form $x = 0$ to $x = 1 - \sum_{k \neq i, \neq j} p_k^{(n}$, one will have a solution to this maximization problem. Recall that the Bayes cost reduction algorithm has been well developed [14], [15]. Taking account of this, one can find such a pair $(\mathbf{p}^*, \Pi^*)$ without any technical difficulty; for instance, one may use the golden section search algorithm together with the Bayes cost reduction algorithm for this part. Once we could find $(\mathbf{p}^*, \Pi^*)$, we set

$$\left\{ \begin{array}{lcl} \Pi^{(n+1} & \leftarrow & \Pi^*, \\ \mathbf{p}^{(n+1} & \leftarrow & \mathbf{p}^*, \\ \bar{P}_{\mathrm{e}}^{(n+1} & \leftarrow & \bar{P}_{\mathrm{e}}(\Pi^{(n+1}, \mathbf{p}^{(n+1)}), \end{array} \right. \tag{30}$$

and proceed to the next stage by choosing another (i, j).

Finally, we check whether the conditions (21)-(23) are satisfied or not, when the increase of $\bar{P}_{\mathrm{e}}^{(\cdot}$ is small enough. If the conditions are satisfied with sufficiently small numerical error, we halt the calculation procedure.

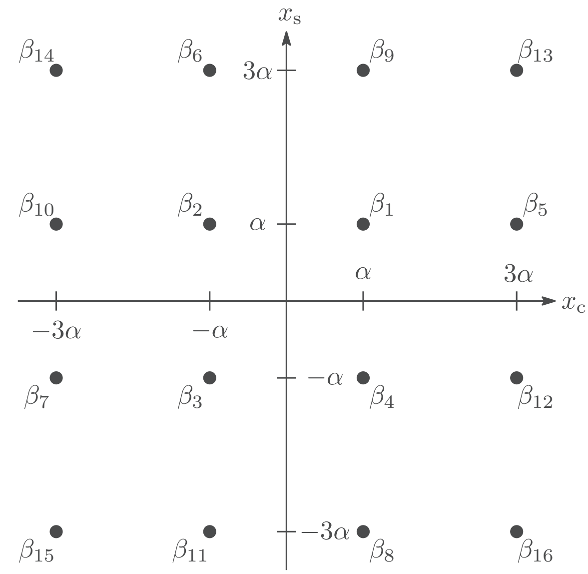

Fig. 1. Signal Constellation of 16QAM

IV. 16QAM coherent state signal

Let $|\psi_i\rangle$ denote the ith signal quantum state. The 16QAM coherent state signal is defined as follows [3]:

$$|\psi_i\rangle = |\beta_i\rangle, \quad 1 \leq i \leq 16, \tag{31}$$

with complex amplitudes

$$\begin{array}{lcllcl} \beta_1 & = & \alpha(+1+\mathrm{i}); & \beta_2 & = & \alpha(-1+\mathrm{i}); \\ \beta_3 & = & \alpha(-1-\mathrm{i}); & \beta_4 & = & \alpha(+1-\mathrm{i}); \\ \beta_5 & = & \alpha(+3+\mathrm{i}); & \beta_6 & = & \alpha(-1+3\mathrm{i}); \\ \beta_7 & = & \alpha(-3-\mathrm{i}); & \beta_8 & = & \alpha(+1-3\mathrm{i}); \\ \beta_9 & = & \alpha(+1+3\mathrm{i}); & \beta_{10} & = & \alpha(-3+\mathrm{i}); \\ \beta_{11} & = & \alpha(-1-3\mathrm{i}); & \beta_{12} & = & \alpha(+3-\mathrm{i}); \\ \beta_{13} & = & \alpha(+3+3\mathrm{i}); & \beta_{14} & = & \alpha(-3+3\mathrm{i}); \\ \beta_{15} & = & \alpha(-3-3\mathrm{i}); & \beta_{16} & = & \alpha(+3-3\mathrm{i}), \end{array} \tag{32}$$

where $|\beta_i\rangle$ stands for the coherent state of light having complex amplitude β_i, and where $\alpha > 0$ and $\mathrm{i} = \sqrt{-1}$. Fig. 1 shows the signal constellation of the 16QAM coherent state signal in the (x_c, x_s)-space. For this signal, we defined the average number of signal photons $\bar{N}_{\mathrm{s}} = 10\alpha^2$ in the literature [3]. At that time, we assumed that the probability distribution of the signal is uniform. However, in the case of the minimax problem, we cannot assign a particular distribution in advance. Hence, we use the parameter α^2 instead of $\bar{N}_{\mathrm{s}}$ to specify the amplitude of each signal in our error performance evaluation.

Before calculating the minimax solution for the 16QAM coherent state signal, we examine the symmetry of the signal constellation in order to reduce the computation time. Let $\mathbf{p}$ be an arbitrarily chosen distribution for this signal:

$$\begin{aligned} \mathbf{p} &= (p_1, p_2, p_3, p_4, \quad p_5, p_6, p_7, p_8, \\ &\qquad p_9, p_{10}, p_{11}, p_{12}, \quad p_{13}, p_{14}, p_{15}, p_{16}) \\ &\equiv \mathbf{p}_{\mathrm{a}}. \end{aligned} \tag{33}$$

From this $\mathbf{p}_{\rm a}$, we form the following distributions.

$$\begin{aligned}\mathbf{p}_{\rm b} &= (p_4, p_3, p_2, p_1, \quad p_{12}, p_{11}, p_{10}, p_9, \\ &\qquad p_8, p_7, p_6, p_5, \quad p_{16}, p_{15}, p_{14}, p_{13}); \quad (34)\\ \mathbf{p}_{\rm c} &= (p_4, p_1, p_2, p_3, \quad p_8, p_5, p_6, p_7, \\ &\qquad p_{12}, p_9, p_{10}, p_{11}, \quad p_{16}, p_{13}, p_{14}, p_{15}); \quad (35)\\ \mathbf{p}_{\rm d} &= (p_3, p_2, p_1, p_4, \quad p_{11}, p_{10}, p_9, p_{12}, \\ &\qquad p_7, p_6, p_5, p_8, \quad p_{15}, p_{14}, p_{13}, p_{16}); \quad (36)\\ \mathbf{p}_{\rm e} &= (p_3, p_4, p_1, p_2, \quad p_7, p_8, p_5, p_6, \\ &\qquad p_{11}, p_{12}, p_9, p_{10}, \quad p_{15}, p_{16}, p_{13}, p_{14}); \quad (37)\\ \mathbf{p}_{\rm f} &= (p_2, p_1, p_4, p_3, \quad p_{10}, p_9, p_{12}, p_{11}, \\ &\qquad p_6, p_5, p_8, p_7, \quad p_{14}, p_{13}, p_{16}, p_{15}); \quad (38)\\ \mathbf{p}_{\rm g} &= (p_2, p_3, p_4, p_1, \quad p_6, p_7, p_8, p_5, \\ &\qquad p_{10}, p_{11}, p_{12}, p_9, \quad p_{14}, p_{15}, p_{16}, p_{13}); \quad (39)\\ \mathbf{p}_{\rm h} &= (p_1, p_4, p_3, p_2, \quad p_9, p_{12}, p_{11}, p_{10}, \\ &\qquad p_5, p_8, p_7, p_6, \quad p_{13}, p_{16}, p_{15}, p_{14}). \quad (40)\end{aligned}$$

From the symmetry of the signal constellation of 16QAM, we obtain

$$\bar{P}_{\rm e}^{\rm bayes}(\mathbf{p}) = \bar{P}_{\rm e}^{\rm bayes}(\mathbf{p}_{\rm a}) = \cdots = \bar{P}_{\rm e}^{\rm bayes}(\mathbf{p}_{\rm h}). \quad (41)$$

By using the concavity of the minimum error probability, we obtain

$$\begin{aligned}\bar{P}_{\rm e}^{\rm bayes}(\mathbf{p}) &= \frac{1}{8}\left(\bar{P}_{\rm e}^{\rm bayes}(\mathbf{p}_{\rm a}) + \cdots + \bar{P}_{\rm e}^{\rm bayes}(\mathbf{p}_{\rm h})\right)\\ &\le \bar{P}_{\rm e}^{\rm bayes}(\mathbf{p}') \quad (42)\end{aligned}$$

with

$$\begin{aligned}\mathbf{p}' &= (\zeta_1, \zeta_1, \zeta_1, \zeta_1, \quad \zeta_2, \zeta_2, \zeta_2, \zeta_2, \\ &\qquad \zeta_2, \zeta_2, \zeta_2, \zeta_2, \quad \zeta_3, \zeta_3, \zeta_3, \zeta_3), \quad (43)\end{aligned}$$

where

$$\begin{aligned}\zeta_1 &= \frac{p_1+p_2+p_3+p_4}{4}; \quad (44)\\ \zeta_2 &= \frac{p_5+p_6+p_7+p_8+p_9+p_{10}+p_{11}+p_{12}}{8}; \quad (45)\\ \zeta_3 &= \frac{p_{13}+p_{14}+p_{15}+p_{16}}{4}. \quad (46)\end{aligned}$$

From Eq.(42), we can expect that

$$\begin{cases} p_1^\circ = p_2^\circ = p_3^\circ = p_4^\circ, \\ p_5^\circ = p_6^\circ = p_7^\circ = p_8^\circ = p_9^\circ = p_{10}^\circ = p_{11}^\circ = p_{12}^\circ, \\ p_{13}^\circ = p_{14}^\circ = p_{15}^\circ = p_{16}^\circ. \end{cases} \quad (47)$$

Our computer simulation is carried out taking into account of this fact.

In the simulation, we calculate the minimax decision rule Π°, the minimax distribution $\mathbf{p}^\circ$, and the minimax value $\bar{P}_{\rm e}^\circ$, together with the minimum error probability $\bar{P}_{\rm e}^{\rm bayes}(\mathbf{u})$ for the uniform distribution $\mathbf{u}$. The error probabilities $\bar{P}_{\rm e}^\circ$ and $\bar{P}_{\rm e}^{\rm bayes}(\mathbf{u})$ are shown in Fig. 2 for the range from $\alpha^2 \sim 0$ to $\alpha^2 = 5.0$. As expected from the definitions of the receivers, we see that $\bar{P}_{\rm e}^\circ > \bar{P}_{\rm e}^{\rm bayes}(\mathbf{u})$ in this figure. Comparing these probabilities with that of the square-root measurement that has been shown in Figure 2-(a) of the literature [3], the behavior of the error probabilities, $\bar{P}_{\rm e}^\circ$, $\bar{P}_{\rm e}^{\rm bayes}(\mathbf{u})$, and $\bar{P}_{\rm e}^{\rm srm}(\mathbf{u})$, are similar to each other.

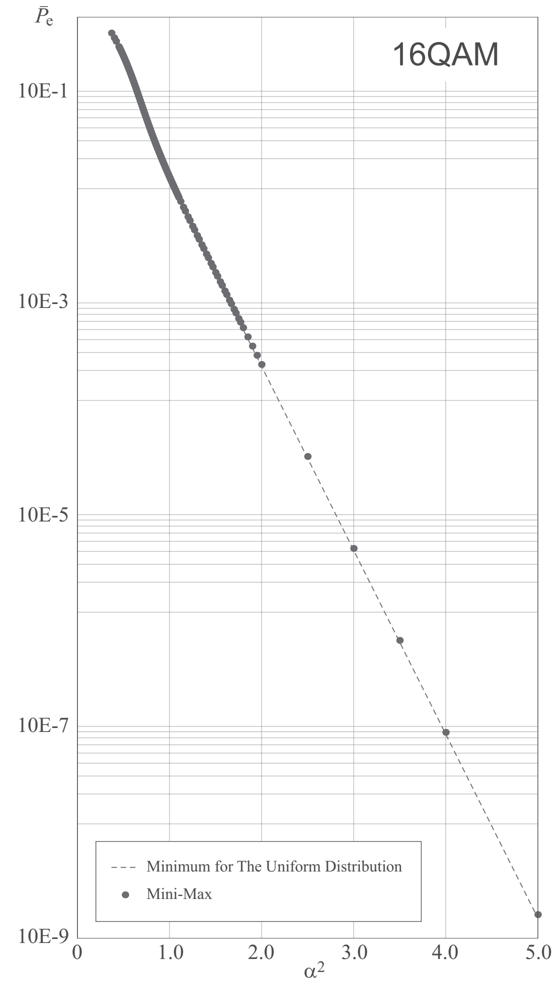

Fig. 2. The minimax value $\bar{P}_{\rm e}^\circ$ and the minimum error probability $\bar{P}_{\rm e}^{\rm bayes}(\mathbf{u})$

Fig. 3 shows an enlarged figure for $\alpha^2 = 0.5$ to $\alpha^2 = 1.0$, together with the corresponding minimax distribution $\mathbf{p}^\circ$. As stated above, the minimax value $\bar{P}_{\rm e}^\circ$ never exceed downward the minimum error probability $\bar{P}_{\rm e}^{\rm bayes}(\mathbf{u})$ for the uniform distribution $\mathbf{u}$, although the difference $\bar{P}_{\rm e}^\circ - \bar{P}_{\rm e}^{\rm bayes}(\mathbf{u})$ is very small. This might lead us to a conclusion that the minimax receiver has less advantage to the Bayes optimal receiver that is designed with the uniform distribution. However, it is not true. To explain the reason why, we treat the case of $\alpha^2 = 1.0$ as an example. In this setting, we examine the average probability of decision errors for the Bayes optimal receiver that is designed with the uniform distribution under the situation of *non-uniform* distributions.

Let $\Pi^{\rm bayes}(\mathbf{u})$ denote the Bayes decision rule for the uniform distribution $\mathbf{u}$. For $\alpha^2 = 1.0$ the diagonal elements of

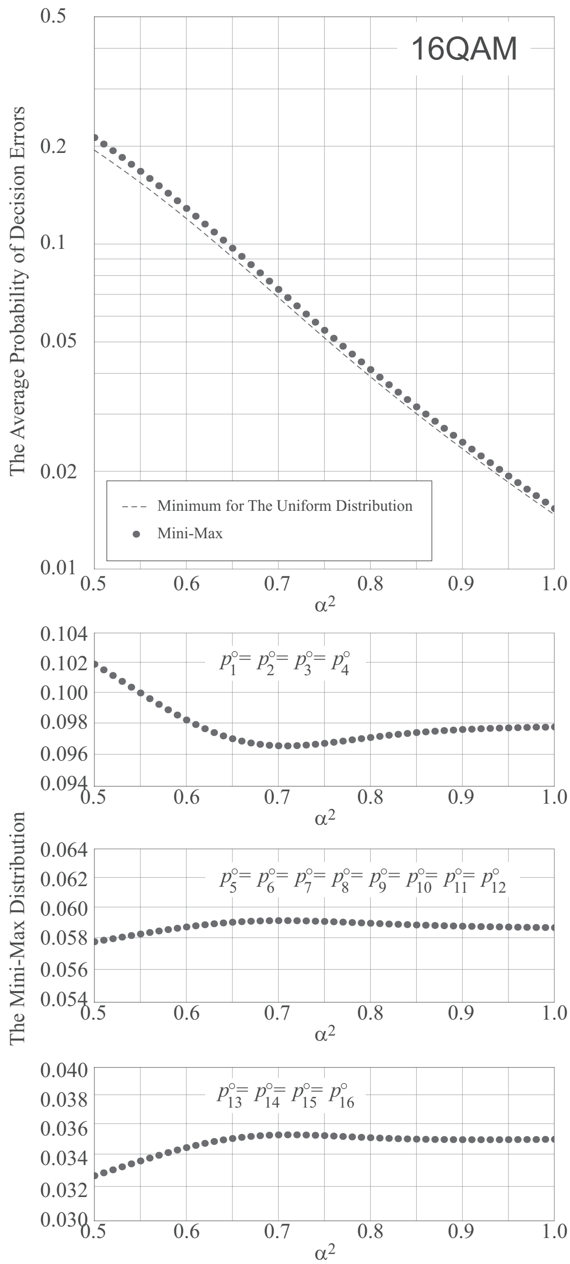

Fig. 3. The minimax value $\bar{P}_{\mathrm{e}}^{\circ}$ and the minimax distribution $\mathbf{p}^{\circ}$.

the channel matrix generated by $\Pi^{\mathrm{bayes}}(\mathbf{u})$ are

$$P^{\mathrm{bayes}}(i|i) = \langle\psi_i|\hat{\Pi}_i^{\mathrm{bayes}}(\mathbf{u})|\psi_i\rangle \simeq \begin{cases} 0.979972, & 1 \le i \le 4; \\ 0.985382, & 5 \le i \le 12; \\ 0.990413, & 13 \le i \le 16, \end{cases} \quad (48)$$

and the average probability of decision errors at the uniform distribution $\mathbf{u}$ is

$$\bar{P}_{\mathrm{e}}^{\mathrm{bayes}}(\mathbf{u}) = \bar{P}_{\mathrm{e}}(\Pi^{\mathrm{bayes}}(\mathbf{u}), \mathbf{u}) \simeq 0.0147126. \quad (49)$$

On the other hand, the minimax receiver has

$$P^{\circ}(i|i) = \langle\psi_i|\hat{\Pi}_i^{\circ}|\psi_i\rangle \simeq 0.984631, \quad 1 \le i \le 16, \quad (50)$$

and the minimax distribution and the minimax value are respectively given as

$$p_i^{\circ} \simeq \begin{cases} 0.0977465, & 1 \le i \le 4; \\ 0.0586488, & 5 \le i \le 12; \\ 0.0349559, & 13 \le i \le 16, \end{cases} \quad (51)$$

and

$$\bar{P}_{\mathrm{e}}^{\circ} = \bar{P}_{\mathrm{e}}(\Pi^{\circ}, \mathbf{p}^{\circ}) \simeq 0.0153696. \quad (52)$$

Since every probability in $\mathbf{p}^{\circ}$ is non-zero, the equality

$$\bar{P}_{\mathrm{e}}(\Pi^{\circ}, \mathbf{q}) = \bar{P}_{\mathrm{e}}^{\circ}, \quad (53)$$

holds for each $\mathbf{q} \in \mathcal{P}$ by Theorem 3. Thus the error performance of the minimax receiver is stable. In other words, when the minimax receiver is used, one can always expect the same performance — $\bar{P}_{\mathrm{e}} \simeq 0.0153696$ — regardless of the probability distribution of the signal. On the other hand, the error performance of the Bayes optimal receiver that is designed with the uniform distribution depends on the probability distribution of the signal. To see this, we consider the following cases for explanation.

Case 1: $\mathbf{q}' \in \mathcal{P}$ with

$$q_i' = \begin{cases} 0.0525, & 1 \le i \le 4; \\ 0.0575, & 5 \le i \le 12; \\ 0.0825, & 13 \le i \le 16. \end{cases} \quad (54)$$

Case 2: $\mathbf{q}'' \in \mathcal{P}$ with

$$q_i'' = \begin{cases} 0.0425, & 1 \le i \le 4; \\ 0.0525, & 5 \le i \le 12; \\ 0.1025, & 13 \le i \le 16. \end{cases} \quad (55)$$

Case 3: $\mathbf{q}''' \in \mathcal{P}$ with

$$q_i''' = \begin{cases} 0.0025, & 1 \le i \le 4; \\ 0.0375, & 5 \le i \le 12; \\ 0.1825, & 13 \le i \le 16. \end{cases} \quad (56)$$

The average probability of decision errors of the Bayes optimal receiver $\Pi^{\mathrm{bayes}}(\mathbf{u})$ for each case is shown in TABLE I. From this table, we can understand that the average probability of decision errors of the Bayes optimal receiver $\Pi^{\mathrm{bayes}}(\mathbf{u})$ is unstable, and it indicates worse performance than the minimax receiver in some region.

V. Conclusion

The error performance of the minimax receiver for the 16QAM coherent state signal has been calculated by using a numerical calculation method that was proposed by the author. In the simulation the average probability of decision errors of the minimax receiver has been compared with that of the Bayes optimal receiver that is designed for the uniform distribution. From the comparison, it has been shown that the performance of the minimax receiver is stable regardless of the probability distribution of the signal, while that of the Bayes optimal

TABLE I
THE BAYES OPTIMAL RECEIVER V.S. THE MINIMAX RECEIVER

Case 1

$$\bar{P}_{\rm e}(\Pi^{\rm bayes}(\mathbf{u}),\mathbf{q}') \simeq 0.015\ 346\ 7 \quad < \bar{P}_{\rm e}^{\circ}$$
$$\bar{P}_{\rm e}(\Pi^{\circ},\mathbf{q}') \simeq 0.015\ 369\ 6 \quad = \bar{P}_{\rm e}^{\circ}$$

Case 2

$$\bar{P}_{\rm e}(\Pi^{\rm bayes}(\mathbf{u}),\mathbf{q}'') \simeq 0.015\ 980\ 7 \quad > \bar{P}_{\rm e}^{\circ}$$
$$\bar{P}_{\rm e}(\Pi^{\circ},\mathbf{q}'') \simeq 0.015\ 369\ 6 \quad = \bar{P}_{\rm e}^{\circ}$$

Case 3

$$\bar{P}_{\rm e}(\Pi^{\rm bayes}(\mathbf{u}),\mathbf{q}''') \simeq 0.018\ 516\ 9 \quad > \bar{P}_{\rm e}^{\circ}$$
$$\bar{P}_{\rm e}(\Pi^{\circ},\mathbf{q}''') \simeq 0.015\ 369\ 6 \quad = \bar{P}_{\rm e}^{\circ}$$

receiver is not. Therefore, the use of the minimax strategy will be a better way than that of the Bayes strategy in designing quantum communication systems.

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