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***Abstract*—This is a note on quantum detection of quaternary amplitude-shift keying (QASK) coherent state signal. The closed-form expression of the square-root measurement (SRM) of QASK coherent state signal is derived by solving the eigenvalue problem of the Gram matrix consisting of the signal. The Bayes-optimal detection and the minimax detection for QASK coherent state signal are respectively analyzed by using novel iterative calculation algorithms of Nakahira *et al.* [15]. Toward derivation of the closed-form expressions of the Bayes-optimal detection and the minimax detection, mathematical structure of the corresponding optimal detection vectors is discussed based on the numerical calculation results.**

I. Introduction

The main role of quantum signal detection theory [1], [2], [3], [4] is to examine the performance limit of quantum state signals and to clarify the mathematical structure of optimal quantum detection. Hence it is expected to be a guiding theory that provides design methods for optimal quantum detection that enables a highly functional communication system which close to quantum limits beyond classical ones. In this article, we attempt to apply quantum signal detection theory to a particular coherent state signal.

A coherent state signal is characterized by its modulation format. For example, phase-shift keying (PSK) and quadrature amplitude modulation (QAM) are major formats widely used in advanced digital coherent optical communication systems [5]. Here let us recall some preceding studies about coherent state signals based on quantum signal detection theory. The early work on the optimal detection problem of PSK coherent state signal can be found in the literatures [6], [7] in which the so-called Belavkin weighted square-root measurement (BWSRM) was introduced. Since PSK coherent state signal is a kind of symmetric quantum state signals, analysis of symmetric signals may involve the case of PSK coherent state signal. In this context, the analysis of optimal detection of symmetric quantum states by Ban *et al.* [8] is remarkable. Based on the analyses of Belavkin and Ban *et al.*, it can be understood that the square-root measurement (SRM) for PSK coherent state signal is not only the Bayes-optimal detection at the uniform signal distribution but also the minimax detection. The numerical comparison of the error rate performance of quantum detection for M-ary PSK and M-ary QAM coherent state signals was done under the assumption that every signal state is pure and the SRM is employed as a receiver [9]. Extending this result, numerical simulations for 4PSK, 8PSK and 16QAM coherent state signals in the presence of thermal noise were performed by Cariolaro and Pierobon [10]. Since the error rate performance of Bayes-optimal detection depends on a priori probability distribution of signal elements in general, the minimax strategy would be preferable than Bayes-optimal strategy in some cases. The error rate performance of the minimax detection for 16QAM coherent state signal was numerically investigated by the author [11].

As for another type of coherent state signals, some results with respect to amplitude-shift keying (ASK) coherent state signal can be found. In the literature [12] by Helstrom, the error rates of Bayes-optimal detection for ternary amplitude-shift keying (3ASK) and quaternary amplitude-shift keying (QASK) coherent state signals were computed using his Bayes-cost reduction algorithm [12]. In contrast, the closed-form expressions of the SRM and the minimax detection for 3ASK coherent state signal were derived by the author [13], [14]. Taking account of recent progress in the applications of quantum signal detection theory, the analysis of M-ary ASK coherent state signal is of importance. However, no comprehensive analysis of M-ary ASK coherent state signal has been done. Towards a future study on M-ary ASK coherent state signal, we focus on quaternary amplitude-shift keying (QASK) coherent state signal in this article as a first step.

As mentioned above, the error rate performance of the Bayes-optimal detection for QASK coherent state signal at the uniform signal distribution has been already shown in the literature [12]. Hence the central problem in our analysis is to derive closed-form expressions of optimal quantum detection in each detection strategy. However, as we will see later, derivation of the closed-form expressions of optimal quantum detection for QASK coherent state signal was restricted only to the case of SRM. Aiming to break through this deadlock, we perform numerical analysis of QASK coherent state signal with the help of the calculation algorithms of Nakahira *et al.* [15], because the results to be obtained in the numerical calculation will be helpful to figure out the structure of optimal quantum detection.

The remaining part of this article is organized as follows. In Section II, the error rate performance and the optimal detection operators of QASK coherent state signal are investigated for the SRM, the Bayes-optimal detection at the uniform signal distribution, and the minimax detection, respectively. In the first half of Section II, the closed-form expression of the SRM for QASK coherent state signal is derived by solving the eigenvalue problem of the corresponding Gram matrix. In the remaining part of Section II, the Bayes-optimal detection and the minimax detection cases are numerically investigated. In each case, the optimal detection vectors and the corresponding minimal average probability of error are numerically shown. In Section III, some discussions on the properties of QASK coherent state signal are given, and we summarize the results in Section IV.

II. Error rate performance of QASK coherent state signal

A. QASK coherent state signal

Let $|\psi_i\rangle$ denote a quantum state that corresponds to a signal element. We define QASK coherent state signal as follows (See also *Appendix A*).

$$\begin{aligned}\mathcal{S} &= \{|\psi_1\rangle, |\psi_2\rangle, |\psi_3\rangle, |\psi_4\rangle\} \\ &= \{|-3\alpha\rangle, |-\alpha\rangle, |\alpha\rangle, |3\alpha\rangle\}, \end{aligned} \tag{1}$$

where $|\alpha\rangle$ is a coherent state defined by $\hat{a}|\alpha\rangle = \alpha|\alpha\rangle$ with the photon annihilation operator $\hat{a}$. In this article, we assume $\alpha > 0$ for simplicity.

The signal constellation of QASK coherent sate signal is shown in Fig. 1, where $\hat{x}_{\rm c} = (\hat{a} + \hat{a}^\dagger)/2$, $\hat{x}_{\rm s} = (\hat{a} - \hat{a}^\dagger)/2{\rm i}$, and ${\rm i} = \sqrt{-1}$. Let $\mathbf{p} = (p_1, p_2, p_3, p_4)$ denote

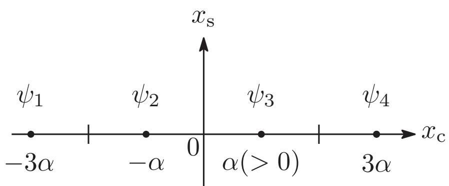

Fig. 1. Signal constellation of QASK.

a probability distribution of the signal elements. If the distribution is uniform, $\mathbf{p} = \mathbf{u} = (1/4, 1/4, 1/4, 1/4)$, then the average number of signal photons for QASK coherent state signal is given as $\bar{n}_{\rm s} = 5|\alpha|^2$.

In general, the Gram operator and the Gram matrix of pure states $|\psi_i\rangle$ are respectively defined by

$$\hat{G} = \sum_{k=1}^{M} |\psi_k\rangle\langle\psi_k|, \tag{2}$$

and

$$\mathsf{G} = \begin{bmatrix} \langle\psi_1|\psi_1\rangle & \langle\psi_1|\psi_2\rangle & \cdots & \langle\psi_1|\psi_M\rangle \\ \langle\psi_2|\psi_1\rangle & \langle\psi_2|\psi_2\rangle & \cdots & \langle\psi_2|\psi_M\rangle \\ \vdots & \vdots & \ddots & \vdots \\ \langle\psi_M|\psi_1\rangle & \langle\psi_M|\psi_2\rangle & \cdots & \langle\psi_M|\psi_M\rangle \end{bmatrix}, \tag{3}$$

where M is the number of states. Suppose that $|\psi_i\rangle$ are linearly independent. Then the Gram operator is strictly positive-definite, $\hat{G} > 0$. The SRM vectors are defined by

$$|\mu_i^\bullet\rangle \equiv \hat{G}^{-1/2}|\psi_i\rangle, \quad 1 \le i \le M. \tag{4}$$

Conversely,

$$|\psi_i\rangle = \hat{G}^{1/2}|\mu_i^\bullet\rangle, \quad 1 \le i \le M. \tag{5}$$

Letting $\gamma = \{|\mu_i^\bullet\rangle : 1 \le i \le M\}$, the set γ is an orthonormal basis for the space spanned by linearly independent states $\{|\psi_i\rangle\}$. In fact, it satisfies

$$\langle\mu_i^\bullet|\mu_j^\bullet\rangle = \delta_{ij}\ \forall(i,j), \quad \text{and} \quad \sum_{k=1}^{M} |\mu_k^\bullet\rangle\langle\mu_k^\bullet| = \hat{1}, \tag{6}$$

where δ_{ij} is the Kronecker delta and $\hat{1}$ is the identity. Note that matrix representation of $\hat{G}$ in the basis γ is the Gram matrix G:

$$\begin{aligned}[\hat{G}]_\gamma &= \left[\langle\mu_i^\bullet|\hat{G}|\mu_j^\bullet\rangle\right] \\ &= \left[\langle\mu_i^\bullet|\hat{G}^{1/2}\hat{G}^{1/2}|\mu_j^\bullet\rangle\right] \\ &= [\langle\psi_i|\psi_j\rangle] = \mathsf{G}. \end{aligned} \tag{7}$$

Applying the completeness relation of Eq.(6) to Eq.(5), we have

$$\begin{aligned}|\psi_i\rangle &= \left(\sum_{k=1}^{M} |\mu_k^\bullet\rangle\langle\mu_k^\bullet|\right) \hat{G}^{1/2}|\mu_i^\bullet\rangle \\ &= \sum_{k=1}^{M} \left[\langle\mu_k^\bullet|\hat{G}^{1/2}|\mu_i^\bullet\rangle\right] |\mu_k^\bullet\rangle \\ &= \sum_{k=1}^{M} \left(\mathsf{g}^{1/2}\right)_{ki} |\mu_k^\bullet\rangle, \end{aligned} \tag{8}$$

where $\left(\mathsf{g}^{1/2}\right)_{ki}$ is the (k,i)-entry of $\mathsf{G}^{1/2}$. From this we observe that the ith column of $\mathsf{G}^{1/2}$ corresponds to the coefficients in the expansion by the basis γ. Conversely, the relation $\hat{G}^{1/2}\hat{G}^{-1/2} = \hat{1}$ and the completeness relation provide the following expression.

$$\begin{aligned}|\mu_i^\bullet\rangle &= \hat{G}^{1/2}\hat{G}^{-1/2}|\mu_i^\bullet\rangle \\ &= \hat{G}^{1/2}\left(\sum_{k=1}^{M} |\mu_k^\bullet\rangle\langle\mu_k^\bullet|\right) \hat{G}^{-1/2}|\mu_i^\bullet\rangle \\ &= \sum_{k=1}^{M} \left[\langle\mu_k^\bullet|\hat{G}^{-1/2}|\mu_i^\bullet\rangle\right] \left(\hat{G}^{1/2}|\mu_k^\bullet\rangle\right) \\ &= \sum_{k=1}^{M} \left(\mathsf{g}^{-1/2}\right)_{ki} |\psi_k\rangle, \end{aligned} \tag{9}$$

where $\left(\mathsf{g}^{-1/2}\right)_{ki}$ is the (k,i)-entry of $\mathsf{G}^{-1/2}$.

From the analysis mentioned above, the column vector representation of each signal element of QASK coherent state signal in the basis γ is given as follows.

$$|\psi_i\rangle \doteq [|\psi_i\rangle]_\gamma = \begin{bmatrix} (\mathsf{g}^{1/2})_{1i} \\ (\mathsf{g}^{1/2})_{2i} \\ (\mathsf{g}^{1/2})_{3i} \\ (\mathsf{g}^{1/2})_{4i} \end{bmatrix}, \quad i=1,2,3,4, \tag{10}$$

where the symbol $\doteq$ means the left-hand side is represented by the right-hand side. Similarly, the column vector representation of each SRM vector for QASK coherent state signal in the basis γ is given by

$$|\mu_i^\bullet\rangle \doteq [|\mu_i^\bullet\rangle]_\gamma = \begin{bmatrix} \delta_{1i} \\ \delta_{2i} \\ \delta_{3i} \\ \delta_{4i} \end{bmatrix}, \quad i=1,2,3,4. \tag{11}$$

B. Square-root of Gram matrix of QASK

The Gram matrix of QASK coherent state signal is given by

$$\mathsf{G} = \begin{bmatrix} 1 & \kappa & \kappa^4 & \kappa^9 \\ \kappa & 1 & \kappa & \kappa^4 \\ \kappa^4 & \kappa & 1 & \kappa \\ \kappa^9 & \kappa^4 & \kappa & 1 \end{bmatrix}, \tag{12}$$

where $\kappa = \exp[-2|\alpha|^2]$. The eigenvalues and the corresponding eigenvectors of G are respectively given as follows:

$$\lambda_1 = \frac{1}{2}(1-\kappa)\left(2-\kappa z_1\sqrt{y_+}\right); \tag{13}$$

$$\lambda_2 = \frac{1}{2}(1+\kappa)\left(2-\kappa z_2\sqrt{y_-}\right); \tag{14}$$

$$\lambda_3 = \frac{1}{2}(1-\kappa)\left(2+\kappa z_3\sqrt{y_+}\right); \tag{15}$$

$$\lambda_4 = \frac{1}{2}(1+\kappa)\left(2+\kappa z_4\sqrt{y_-}\right), \tag{16}$$

and

$$\vec{\lambda}_1 = \frac{1}{2}\begin{bmatrix} \sqrt{z_1} \\ -\sqrt{z_3} \\ \sqrt{z_3} \\ -\sqrt{z_1} \end{bmatrix}; \tag{17}$$

$$\vec{\lambda}_2 = \frac{1}{2}\begin{bmatrix} \sqrt{z_2} \\ -\sqrt{z_4} \\ -\sqrt{z_4} \\ \sqrt{z_2} \end{bmatrix}; \tag{18}$$

$$\vec{\lambda}_3 = \frac{1}{2}\begin{bmatrix} \sqrt{z_3} \\ \sqrt{z_1} \\ -\sqrt{z_1} \\ -\sqrt{z_3} \end{bmatrix}; \tag{19}$$

$$\vec{\lambda}_4 = \frac{1}{2}\begin{bmatrix} \sqrt{z_4} \\ \sqrt{z_2} \\ \sqrt{z_2} \\ \sqrt{z_4} \end{bmatrix}, \tag{20}$$

where

$$z_1 = 1-\frac{x_+}{\sqrt{y_+}}; \tag{21}$$

$$z_2 = 1+\frac{x_-}{\sqrt{y_-}}; \tag{22}$$

$$z_3 = 1+\frac{x_+}{\sqrt{y_+}}; \tag{23}$$

$$z_4 = 1-\frac{x_-}{\sqrt{y_-}}, \tag{24}$$

and

$$y_+ = 4\left(1+\kappa+\kappa^2\right)^2 + x_+^2; \tag{25}$$

$$y_- = 4\left(1-\kappa+\kappa^2\right)^2 + x_-^2; \tag{26}$$

$$x_+ = (1+\kappa)(1+\kappa^2)(1+\kappa^4); \tag{27}$$

$$x_- = (1-\kappa)(1+\kappa^2)(1+\kappa^4), \tag{28}$$

and where the eigenvectors have been normalized. Since the signal elements are linearly independent, the Gram matrix G of Eq.(12) is a positive definite matrix. Therefore every eigenvalue is positive, $\lambda_i > 0,\ i=1,2,3,4$.

Further, the projectors $\mathsf{P}_i = \vec{\lambda}_i {}^t\vec{\lambda}_i$, where ${}^t\vec{\lambda}_i$ means the transpose of $\vec{\lambda}_i$, are given by

$$\mathsf{P}_1 = \frac{1}{4}\begin{bmatrix} z_1 & -\sqrt{z_1z_3} & \sqrt{z_1z_3} & -z_1 \\ -\sqrt{z_1z_3} & z_3 & -z_3 & \sqrt{z_1z_3} \\ \sqrt{z_1z_3} & -z_3 & z_3 & -\sqrt{z_1z_3} \\ -z_1 & \sqrt{z_1z_3} & -\sqrt{z_1z_3} & z_1 \end{bmatrix}, \tag{29}$$

$$\mathsf{P}_2 = \frac{1}{4}\begin{bmatrix} z_2 & -\sqrt{z_2z_4} & -\sqrt{z_2z_4} & z_2 \\ -\sqrt{z_2z_4} & z_4 & z_4 & -\sqrt{z_2z_4} \\ -\sqrt{z_2z_4} & z_4 & z_4 & -\sqrt{z_3z_4} \\ z_2 & -\sqrt{z_2z_4} & -\sqrt{z_2z_4} & z_2 \end{bmatrix}, \tag{30}$$

$$\mathsf{P}_3 = \frac{1}{4}\begin{bmatrix} z_3 & \sqrt{z_1z_3} & -\sqrt{z_1z_3} & -z_3 \\ \sqrt{z_1z_3} & z_1 & -z_1 & -\sqrt{z_1z_3} \\ -\sqrt{z_1z_3} & -z_1 & z_1 & \sqrt{z_1z_3} \\ -z_3 & -\sqrt{z_1z_3} & \sqrt{z_1z_3} & z_3 \end{bmatrix}, \tag{31}$$

$$\mathsf{P}_4 = \frac{1}{4}\begin{bmatrix} z_4 & \sqrt{z_2z_4} & \sqrt{z_2z_4} & z_4 \\ \sqrt{z_2z_4} & z_2 & z_2 & \sqrt{z_2z_4} \\ \sqrt{z_2z_4} & z_2 & z_2 & \sqrt{z_3z_4} \\ z_4 & \sqrt{z_2z_4} & \sqrt{z_2z_4} & z_4 \end{bmatrix}. \tag{32}$$

A straightforward calculation yields the following properties:

1) Eigenequation holds:

$$\mathsf{G}\vec{\lambda}_i = \lambda_i\vec{\lambda}_i,\ i=1,2,3,4.$$

2) Trace of Gram matrix:

$$\lambda_1+\lambda_2+\lambda_3+\lambda_4 = 4.$$

3) Orthonormality of eigenvectors:

$$ {}^t\vec{\lambda}_i\vec{\lambda}_j = \delta_{ij},\ (i,j) \in \{1,2,3,4\}^2. $$

4) Completeness of eigenvectors:

$$ \mathsf{P}_1 + \mathsf{P}_2 + \mathsf{P}_3 + \mathsf{P}_4 = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}. $$

5) Reconstruction of Gram matrix:

$$ \lambda_1\mathsf{P}_1 + \lambda_2\mathsf{P}_2 + \lambda_3\mathsf{P}_3 + \lambda_4\mathsf{P}_4 = \mathsf{G}. $$

By using the eigenvalues λ_i and the projectors P_i, $\mathsf{G}^{1/2}$ is formally given by

$$ \begin{aligned} \mathsf{G}^{1/2} &= \sqrt{\lambda_1}\mathsf{P}_1 + \sqrt{\lambda_2}\mathsf{P}_2 + \sqrt{\lambda_3}\mathsf{P}_3 + \sqrt{\lambda_4}\mathsf{P}_4 \\ &= \begin{bmatrix} (\mathsf{g}^{1/2})_{11} & (\mathsf{g}^{1/2})_{12} & (\mathsf{g}^{1/2})_{13} & (\mathsf{g}^{1/2})_{14} \\ (\mathsf{g}^{1/2})_{21} & (\mathsf{g}^{1/2})_{22} & (\mathsf{g}^{1/2})_{23} & (\mathsf{g}^{1/2})_{24} \\ (\mathsf{g}^{1/2})_{31} & (\mathsf{g}^{1/2})_{32} & (\mathsf{g}^{1/2})_{33} & (\mathsf{g}^{1/2})_{34} \\ (\mathsf{g}^{1/2})_{41} & (\mathsf{g}^{1/2})_{42} & (\mathsf{g}^{1/2})_{43} & (\mathsf{g}^{1/2})_{44} \end{bmatrix}. \end{aligned} \tag{33} $$

Substituting Eqs.(29)-(32) to this matrix, each entry of this matrix is given as follows:

$$ \begin{aligned} (\mathsf{g}^{1/2})_{11} &= \frac{1}{4}\left(\sqrt{\lambda_1}z_1 + \sqrt{\lambda_2}z_2 + \sqrt{\lambda_3}z_3 + \sqrt{\lambda_4}z_4\right) \\ &= (\mathsf{g}^{1/2})_{44}; \end{aligned} \tag{34} $$

$$ \begin{aligned} (\mathsf{g}^{1/2})_{12} &= \frac{1}{4}\Big\{\left(-\sqrt{\lambda_1} + \sqrt{\lambda_3}\right)\sqrt{z_1z_3} \\ &\qquad + \left(\sqrt{\lambda_2} + \sqrt{\lambda_4}\right)\sqrt{z_2z_4}\Big\} \\ &= (\mathsf{g}^{1/2})_{21} = (\mathsf{g}^{1/2})_{34} = (\mathsf{g}^{1/2})_{43}; \end{aligned} \tag{35} $$

$$ \begin{aligned} (\mathsf{g}^{1/2})_{13} &= \frac{1}{4}\Big\{\left(\sqrt{\lambda_1} - \sqrt{\lambda_3}\right)\sqrt{z_1z_3} \\ &\qquad + \left(-\sqrt{\lambda_2} + \sqrt{\lambda_4}\right)\sqrt{z_2z_4}\Big\} \\ &= (\mathsf{g}^{1/2})_{31} = (\mathsf{g}^{1/2})_{24} = (\mathsf{g}^{1/2})_{42}; \end{aligned} \tag{36} $$

$$ \begin{aligned} (\mathsf{g}^{1/2})_{14} &= \frac{1}{4}\left(-\sqrt{\lambda_1}z_1 + \sqrt{\lambda_2}z_2 - \sqrt{\lambda_3}z_3 + \sqrt{\lambda_4}z_4\right) \\ &= (\mathsf{g}^{1/2})_{41}; \end{aligned} \tag{37} $$

$$ \begin{aligned} (\mathsf{g}^{1/2})_{22} &= \frac{1}{4}\left(\sqrt{\lambda_3}z_1 + \sqrt{\lambda_4}z_2 + \sqrt{\lambda_1}z_3 + \sqrt{\lambda_2}z_4\right) \\ &= (\mathsf{g}^{1/2})_{33}; \end{aligned} \tag{38} $$

$$ \begin{aligned} (\mathsf{g}^{1/2})_{23} &= \frac{1}{4}\left(-\sqrt{\lambda_3}z_1 + \sqrt{\lambda_4}z_2 - \sqrt{\lambda_1}z_3 + \sqrt{\lambda_2}z_4\right) \\ &= (\mathsf{g}^{1/2})_{32}. \end{aligned} \tag{39} $$

Similarly, we have the inverse of the square-root of Gram matrix as follows:

$$ \begin{aligned} &\mathsf{G}^{-1/2} \\ &= \frac{1}{\sqrt{\lambda_1}}\mathsf{P}_1 + \frac{1}{\sqrt{\lambda_2}}\mathsf{P}_2 + \frac{1}{\sqrt{\lambda_3}}\mathsf{P}_3 + \frac{1}{\sqrt{\lambda_4}}\mathsf{P}_4 \\ &= \begin{bmatrix} (\mathsf{g}^{-1/2})_{11} & (\mathsf{g}^{-1/2})_{12} & (\mathsf{g}^{-1/2})_{13} & (\mathsf{g}^{-1/2})_{14} \\ (\mathsf{g}^{-1/2})_{21} & (\mathsf{g}^{-1/2})_{22} & (\mathsf{g}^{-1/2})_{23} & (\mathsf{g}^{-1/2})_{24} \\ (\mathsf{g}^{-1/2})_{31} & (\mathsf{g}^{-1/2})_{32} & (\mathsf{g}^{-1/2})_{33} & (\mathsf{g}^{-1/2})_{34} \\ (\mathsf{g}^{-1/2})_{41} & (\mathsf{g}^{-1/2})_{42} & (\mathsf{g}^{-1/2})_{43} & (\mathsf{g}^{-1/2})_{44} \end{bmatrix}, \end{aligned} \tag{40} $$

where

$$ \begin{aligned} (\mathsf{g}^{-1/2})_{11} &= \frac{1}{4}\left(\frac{z_1}{\sqrt{\lambda_1}} + \frac{z_2}{\sqrt{\lambda_2}} + \frac{z_3}{\sqrt{\lambda_3}} + \frac{z_4}{\sqrt{\lambda_4}}\right) \\ &= (\mathsf{g}^{-1/2})_{44}; \end{aligned} \tag{41} $$

$$ \begin{aligned} (\mathsf{g}^{-1/2})_{12} &= \frac{1}{4}\Big\{\left(-\frac{1}{\sqrt{\lambda_1}} + \frac{1}{\sqrt{\lambda_3}}\right)\sqrt{z_1z_3} \\ &\qquad + \left(\frac{1}{\sqrt{\lambda_2}} + \frac{1}{\sqrt{\lambda_4}}\right)\sqrt{z_2z_4}\Big\} \\ &= (\mathsf{g}^{-1/2})_{21} = (\mathsf{g}^{-1/2})_{34} = (\mathsf{g}^{-1/2})_{43}; \end{aligned} \tag{42} $$

$$ \begin{aligned} (\mathsf{g}^{-1/2})_{13} &= \frac{1}{4}\Big\{\left(\frac{1}{\sqrt{\lambda_1}} - \frac{1}{\sqrt{\lambda_3}}\right)\sqrt{z_1z_3} \\ &\qquad + \left(-\frac{1}{\sqrt{\lambda_2}} + \frac{1}{\sqrt{\lambda_4}}\right)\sqrt{z_2z_4}\Big\} \\ &= (\mathsf{g}^{-1/2})_{31} = (\mathsf{g}^{-1/2})_{24} = (\mathsf{g}^{-1/2})_{42}; \end{aligned} \tag{43} $$

$$ \begin{aligned} (\mathsf{g}^{-1/2})_{14} &= \frac{1}{4}\left(-\frac{z_1}{\sqrt{\lambda_1}} + \frac{z_2}{\sqrt{\lambda_2}} - \frac{z_3}{\sqrt{\lambda_3}} + \frac{z_4}{\sqrt{\lambda_4}}\right) \\ &= (\mathsf{g}^{-1/2})_{41}; \end{aligned} \tag{44} $$

$$ \begin{aligned} (\mathsf{g}^{-1/2})_{22} &= \frac{1}{4}\left(\frac{z_1}{\sqrt{\lambda_3}} + \frac{z_2}{\sqrt{\lambda_4}} + \frac{z_3}{\sqrt{\lambda_1}} + \frac{z_4}{\sqrt{\lambda_2}}\right) \\ &= (\mathsf{g}^{-1/2})_{33}; \end{aligned} \tag{45} $$

$$ \begin{aligned} (\mathsf{g}^{-1/2})_{23} &= \frac{1}{4}\left(-\frac{z_1}{\sqrt{\lambda_3}} + \frac{z_2}{\sqrt{\lambda_4}} - \frac{z_3}{\sqrt{\lambda_1}} + \frac{z_4}{\sqrt{\lambda_2}}\right) \\ &= (\mathsf{g}^{-1/2})_{32}. \end{aligned} \tag{46} $$

Some specific examples of the square-root of the Gram matrix for QASK coherent state signal are shown in *Appendix B*. Further, we show some examples of the column vector representation of the signal elements in *Appendix C* for the later analysis done in Section III.

C. SRM for QASK

The detection vectors of the SRM are defined by $|\mu_i^\bullet\rangle = \hat{G}^{-1/2}|\psi_i\rangle$. The column vector representation of $|\mu_i^\bullet\rangle$ for QASK coherent state signal in the basis γ was already shown in Eq.(11). In accordance with Theorem 5 of the literature [16], we set $\mathbf{p}^\bullet = (p_1^\bullet, p_2^\bullet, p_3^\bullet, p_4^\bullet)$ with

$$p_1^\bullet = p_4^\bullet \quad = \quad \frac{(\mathrm{g}^{1/2})_{22}}{2\left\{(\mathrm{g}^{1/2})_{11} + (\mathrm{g}^{1/2})_{22}\right\}}, \qquad (47)$$

$$p_2^\bullet = p_3^\bullet \quad = \quad \frac{(\mathrm{g}^{1/2})_{11}}{2\left\{(\mathrm{g}^{1/2})_{11} + (\mathrm{g}^{1/2})_{22}\right\}}. \qquad (48)$$

It satisfies

$$|\mu_i^\bullet\rangle\langle\mu_i^\bullet|\Big(p_i^\bullet|\psi_i\rangle\langle\psi_i| - p_j^\bullet|\psi_j\rangle\langle\psi_j|\Big)|\mu_j^\bullet\rangle\langle\mu_j^\bullet| = 0 \quad (49)$$

for every (i, j). Therefore, $\mathit{\Pi}^\bullet = \{|\mu_i^\bullet\rangle\langle\mu_i^\bullet| : i = 1, 2, 3, 4\}$ becomes the Bayes-optimal detection strategy at the signal distribution $\mathbf{p}^\bullet$, and the closed-form expression of the minimal average probability of error at $\mathbf{p}^\bullet$ is given by

$$\begin{aligned} \bar{P}_\mathrm{e}^\bullet \quad &= \quad \min_{\mathit{\Pi}} \bar{P}_\mathrm{e}(\mathit{\Pi}, \mathbf{p}^\bullet) \\ &= \quad \bar{P}_\mathrm{e}(\mathit{\Pi}^\bullet, \mathbf{p}^\bullet) = 1 - (\mathrm{g}^{1/2})_{11}(\mathrm{g}^{1/2})_{22}. \quad (50) \end{aligned}$$

This minimal error probability $\bar{P}_\mathrm{e}^\bullet$ at $\mathbf{p}^\bullet$ is illustrated in (a) of Fig. 2. The associated signal probabilities, $p_1^\bullet = p_4^\bullet$ and $p_2^\bullet = p_3^\bullet$, are plotted in (b) and (c) of Fig. 2, respectively. In each figure, the parameter κ was taken from 0.001 to 0.99. Some specific examples of the optimal distribution $\mathbf{p}^\bullet$, the channel matrix obtained by the SRM, and the minimal error probability $\bar{P}_\mathrm{e}^\bullet$ for $\kappa = 0.1, 0.3, 0.5, 0.7$, and 0.9 are shown in *Appendix D*.

D. Bayes-optimal detection of QASK at the uniform input

Numerical analysis of error rate performance of the Bayes-optimal detection for QASK coherent state signal at the uniform signal distribution can be found in Fig.1 of the literature [12] by Helstrom. In this section, we redo a numerical simulation in the same problem settings as that but by using another algorithm.

The problem is to find the minimal error probability $\bar{P}_\mathrm{e}^\mathrm{bayes}(\mathbf{u})$ such that

$$\bar{P}_\mathrm{e}^\mathrm{bayes}(\mathbf{u}) = \min_{\mathit{\Pi}} \bar{P}_\mathrm{e}(\mathit{\Pi}, \mathbf{u}), \qquad (51)$$

where $\mathit{\Pi}$ stands for a positive operator-valued measure (POVM). For this type of optimization problem, several numerical calculation algorithms have been developed [12], [17], [18], [15]. In this article, we use Nakahira's iterative algorithm for finding the Bayes-optimal error probability (Section IV.A of the literature [15]). Calculation program was implemented by *Mathematica*, and the constant for stopping criteria in Nakahira's algorithm was set to be $\delta P_\mathrm{C} = 10^{-12}$. The optimality of the simulation results has been verified with the condition (15) of the literature [15] which is equivalent to the Holevo's original condition for Bayes-optimality [2].

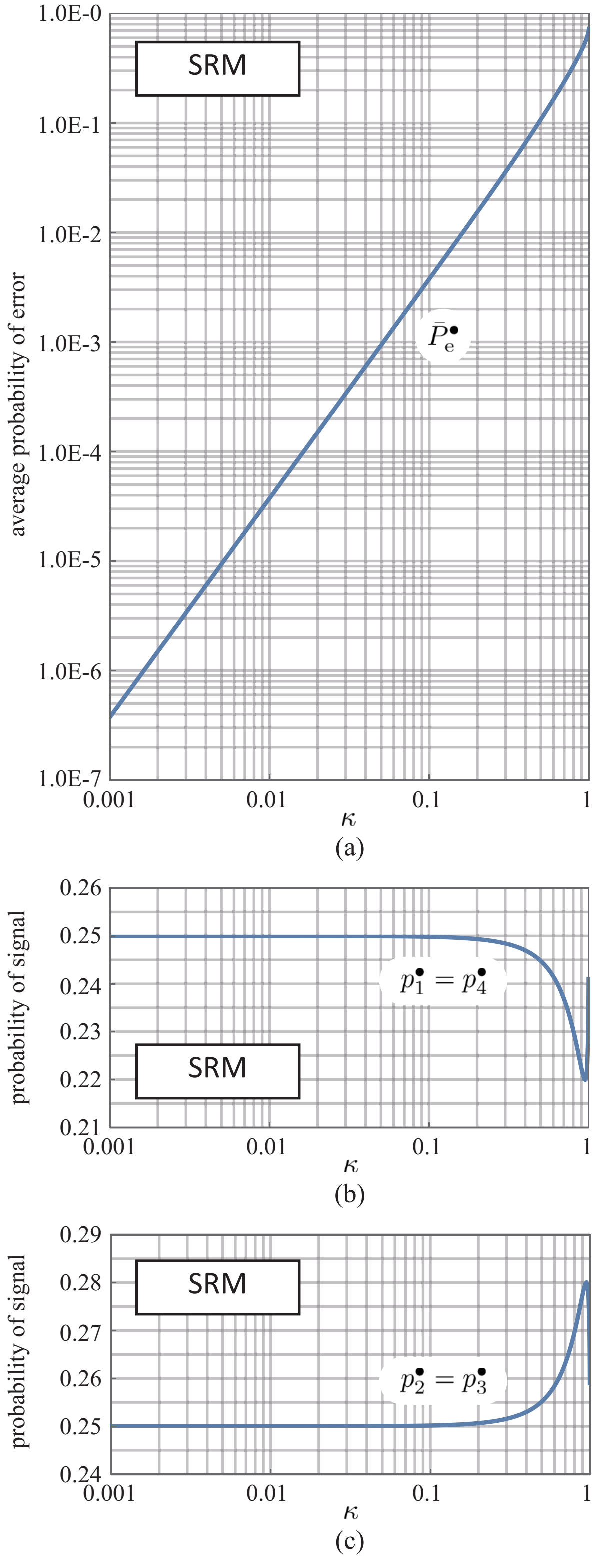

Fig. 2. SRM: (a) $\bar{P}_\mathrm{e}^\bullet$, (b) $p_1^\bullet = p_4^\bullet$, and (c) $p_2^\bullet = p_3^\bullet$.

The simulation results are shown in Fig. 3, which is essentially the same as Fig.1 of the literature [12] although the parameters of horizontal axis are different.

The parameter κ in Fig. 3 was taken from 0.001 to 0.99. Some specific examples of the Bayes-optimal detection vectors $|\mu_i^{\mathrm{bayes}}(\mathbf{u})\rangle$, the channel matrix obtained by the Bayes-optimal detection, and the minimal error probability $\bar{P}_{\mathrm{e}}^{\mathrm{bayes}}(\mathbf{u})$ for $\kappa = 0.1, 0.3, 0.5, 0.7$, and 0.9 are shown in *Appendix E*.

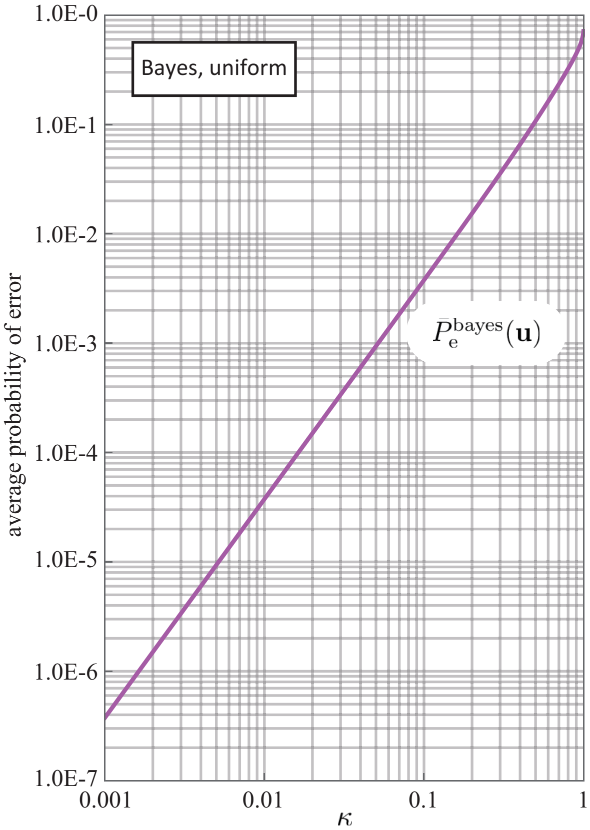

Fig. 3. Bayes-optimal detection at $\mathbf{u}$: $\bar{P}_{\mathrm{e}}^{\mathrm{bayes}}(\mathbf{u})$.

E. Minimax detection of QASK

The minimax strategy is one of fundamental strategies in quantum signal detection theory [4], [19], [20]. The minimax detection problem is formulated as follows:

$$\begin{aligned} \bar{P}_{\mathrm{e}}^{\circ} &= \bar{P}_{\mathrm{e}}(\Pi^{\circ}, \mathbf{p}^{\circ}) \\ &= \min_{\Pi} \max_{\mathbf{p}} \bar{P}_{\mathrm{e}}(\Pi, \mathbf{p}) \\ &= \max_{\mathbf{p}} \min_{\Pi} \bar{P}_{\mathrm{e}}(\Pi, \mathbf{p}). \end{aligned} \quad (52)$$

To solve this problem, Nakahira's iterative calculation algorithm for the minimax detection problem (Section V.A of the literature [15]) is used. Like in the Bayes-optimal detection case, calculation program was implemented by *Mathematica* and the stopping constant was set to $\delta P_{\mathrm{C}} = 10^{-12}$. The simulation results have been verified with the condition (22) of the literature [15] which is equivalent to the optimality conditions of Hirota and Ikehara [4]. The simulation results are shown in Fig. 4. Note that numerically obtained result on the optimal probability p_1° (or p_2°) is equal to that of p_4° (p_3°). This is a reflection of its own symmetric structure of QASK coherent state signal. Some specific examples of the minimax distribution $\mathbf{p}^{\circ}$, the minimax detection vectors $|\mu_i^{\circ}\rangle$, the channel matrix, and the minimax value $\bar{P}_{\mathrm{e}}^{\circ}$ for $\kappa = 0.1, 0.3, 0.5, 0.7$, and 0.9 are shown in *Appendix F*.

III. Some discussions on QASK coherent state signal

A. Comparison of SRM, Bayes-optimal, minimax

Here let us compare the three cases considered in the preceding section. Basically, the following discussion is parallel to Section IV of the literature [14].

To begin with, we define the following factors.

- The rate of difference for $\bar{P}_{\mathrm{e}}$:
$$\epsilon(\bar{P}_{\mathrm{e}}) = \frac{\bar{P}_{\mathrm{e}}^{\circ} - \bar{P}_{\mathrm{e}}^{\bullet}}{\bar{P}_{\mathrm{e}}^{\bullet}} \text{ and } \epsilon'(\bar{P}_{\mathrm{e}}) = \frac{\bar{P}_{\mathrm{e}}^{\mathrm{bayes}}(\mathbf{u}) - \bar{P}_{\mathrm{e}}^{\bullet}}{\bar{P}_{\mathrm{e}}^{\bullet}}.$$
- The rate of difference for p_1:
$$\epsilon(p_1) = \frac{p_1^{\circ} - p_1^{\bullet}}{p_1^{\bullet}} \text{ and } \epsilon'(p_1) = \frac{1/4 - p_1^{\bullet}}{p_1^{\bullet}}.$$
- The rate of difference for p_2:
$$\epsilon(p_2) = \frac{p_2^{\circ} - p_2^{\bullet}}{p_2^{\bullet}} \text{ and } \epsilon'(p_2) = \frac{1/4 - p_2^{\bullet}}{p_2^{\bullet}}.$$

The behavior of these factors is shown in Fig.5. In each comparison, the parameter κ was taken from 0.001 to 0.99. Fig. 5 (a) shows $\epsilon(\bar{P}_{\mathrm{e}}) > 0.0$ and $\epsilon'(\bar{P}_{\mathrm{e}}) < 0.0$. That is,

$$\bar{P}_{\mathrm{e}}^{\circ} > \bar{P}_{\mathrm{e}}^{\bullet} > \bar{P}_{\mathrm{e}}^{\mathrm{bayes}}(\mathbf{u}) \quad (53)$$

holds for $0.001 \le \kappa \le 0.99$. The similar relation was observed also in the case of 3ASK [14]. Similarly, we observed $\epsilon(p_1) < 0.0$ and $\epsilon'(p_1) > 0.0$ in (b) of Fig. 5. Hence we can say that

$$p_1^{\circ} < p_1^{\bullet} < 0.25 \quad (54)$$

holds for $0.001 \le \kappa \le 0.99$. Alternatively,

$$p_2^{\circ} > p_2^{\bullet} > 0.25 \quad (55)$$

holds for $0.001 \le \kappa \le 0.99$, because $\epsilon(p_2) > 0.0$ and $\epsilon'(p_2) < 0.0$ are observed in (c) of Fig. 5. Overviewing (a), (b), and (c) of Fig. 5 and the specific examples for $\kappa = 0.1$, we observed that the structure of the detection vectors of the SRM is similar to that of the Bayes-optimal detection at the uniform distribution of signal elements when $\kappa < 0.1$. On the other hand, all the optimal probability of signal converges to 1/4 when κ is close to 1, but this is just a reflection of pure guessing situation caused by the preparation of almost identical quantum states.

A typical schematic of the relationships (53)-(55) is shown in Fig. 6, where we have assumed $p_1 = p_4$

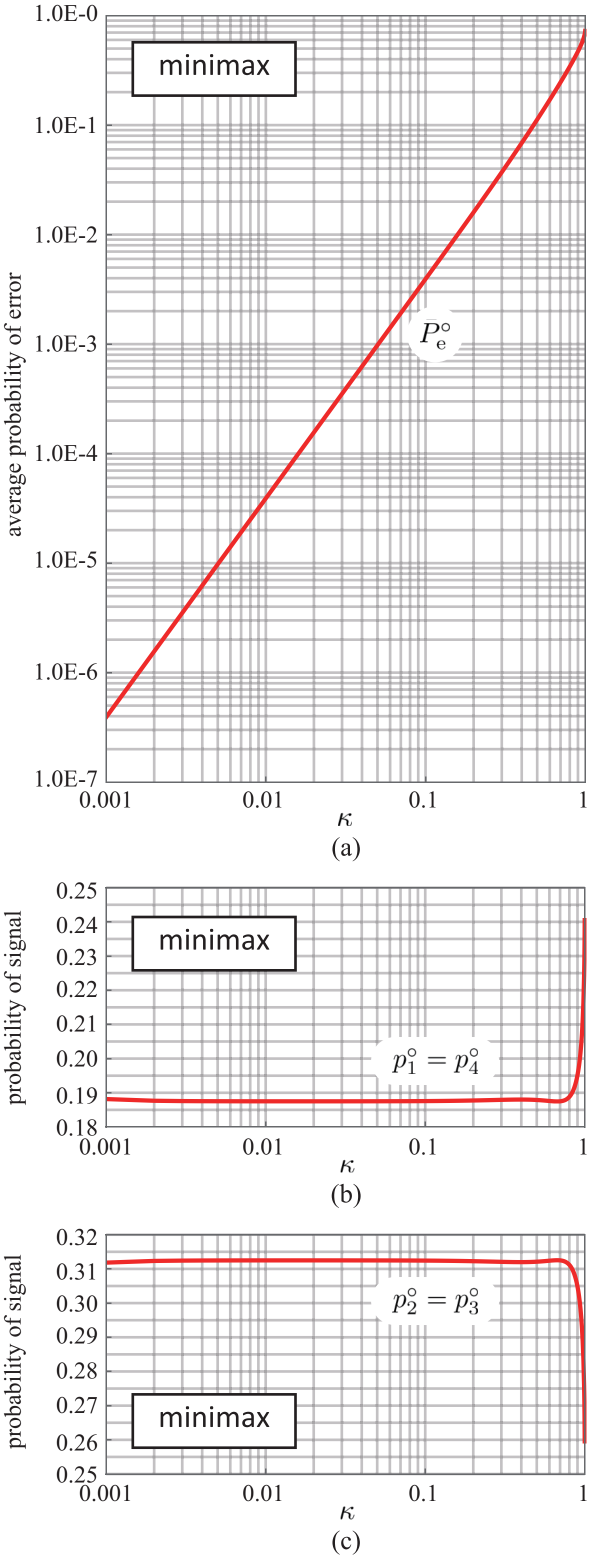

Fig. 4. Minimax detection: (a) $\bar{P}_e^\circ$, (b) $p_1^\circ = p_4^\circ$, and (c) $p_2^\circ = p_3^\circ$.

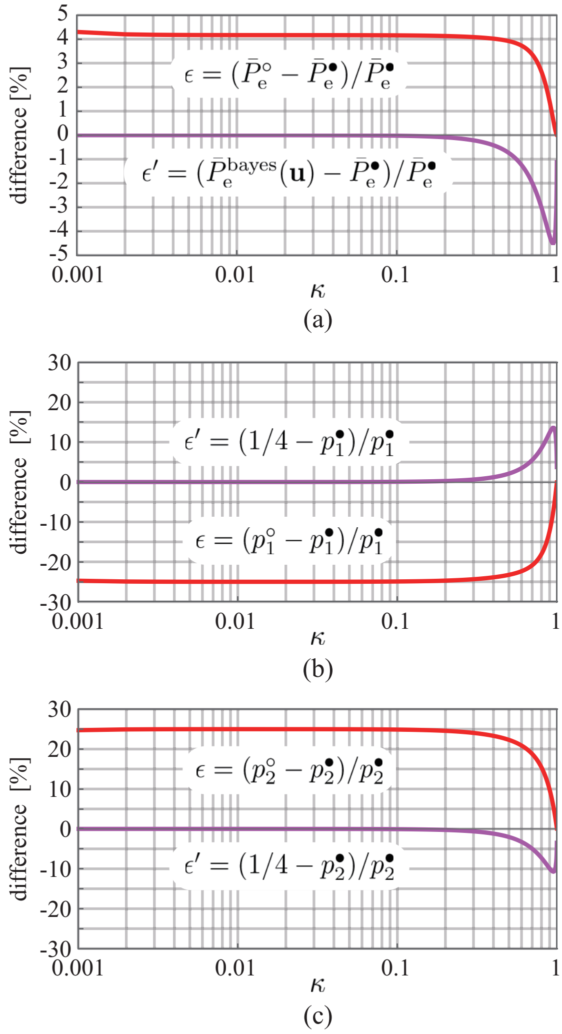

Fig. 5. Rates of difference. (a) $\bar{P}_e$, (b) p_1 (or p_4), (c) p_2 (or p_3).

and $p_2 = p_3$ for simplicity. In this figure, the concave curve stands for the all of Bayes-optimal detection cases, $\bar{P}_e^{\text{bayes}}(\mathbf{p})$. Point $\mathsf{A}(0.23690, 0.23820)$ stands for the case of SRM, Point $\mathsf{B}(0.18750, 0.24621)$ the case of minimax detection, and Point $\mathsf{C}(0.25000, 0.23348)$ the case of Bayes-optimum detection at the uniform distribution of signal elements. The straight lines (dashed, dotted and dot-dashed) are tangent lines that touch to the concave curve at Points A, B, and C, respectively. Each line represents the error rate performance of the corresponding detection strategy when the probability distribution of signal elements varies from the optimal one. The error rate performance of the minimax detection is stable even if the distribution varies. This is a basic feature of the minimax detection.

B. Toward derivation of the optimal detection vectors

In the analysis of error rate performance for the Bayes-optimal detection and the minimax detection, we performed a numerical calculation. Hence the closed-form solutions are still open. Toward analytical derivation of the optimal detection vectors in each case, we analyze the structure of the vectors based on the examples shown in

Sections II.*D* and II.*E*. From observations of the results, a template of the optimal detection vectors is conjectured to the following form.

$$\left[|\mu_1^{\text{bayes}}(\mathbf{u})\rangle\right]_\gamma, \text{ or } [|\mu_1^\circ\rangle]_\gamma \quad \to \quad \begin{bmatrix} A \\ B \\ C \\ D \end{bmatrix}; \tag{56}$$

$$\left[|\mu_2^{\text{bayes}}(\mathbf{u})\rangle\right]_\gamma, \text{ or } [|\mu_2^\circ\rangle]_\gamma \quad \to \quad \begin{bmatrix} -B \\ A \\ D \\ -C \end{bmatrix}; \tag{57}$$

$$\left[|\mu_3^{\text{bayes}}(\mathbf{u})\rangle\right]_\gamma, \text{ or } [|\mu_3^\circ\rangle]_\gamma \quad \to \quad \begin{bmatrix} -C \\ D \\ A \\ -B \end{bmatrix}; \tag{58}$$

$$\left[|\mu_4^{\text{bayes}}(\mathbf{u})\rangle\right]_\gamma, \text{ or } [|\mu_4^\circ\rangle]_\gamma \quad \to \quad \begin{bmatrix} D \\ C \\ B \\ A \end{bmatrix}. \tag{59}$$

As demonstrated in the literatures [4], [21], this type of conjecture might be helpful to derive the closed-form expressions. If these are the optimal detection vectors, then the parameters A, B, C, and D must satisfy the orthonormality condition,

$$A^2 + B^2 + C^2 + D^2 = 1, \quad AD + BC = 0. \tag{60}$$

In the case of Bayes-optimal detection at the uniform distribution of signal elements, its optimality condition yields the following equations.

$$\begin{aligned}
&(r_1 A + r_2 B + r_3 C + r_4 D) \\
&\quad \times (r_2 A - r_1 B - r_4 C + r_3 D) \\
&\qquad = (r_2 A + r_5 B + r_6 C + r_3 D) \\
&\qquad\quad \times (r_5 A - r_2 B - r_3 C + r_6 D); \qquad (61) \\
&(r_1 A + r_2 B + r_3 C + r_4 D) \\
&\quad \times (r_3 A - r_4 B - r_1 C + r_2 D) \\
&\qquad = (r_3 A + r_6 B + r_5 C + r_2 D) \\
&\qquad\quad \times (r_5 A - r_2 B - r_3 C + r_6 D), \qquad (62)
\end{aligned}$$

where $r_1 = (\mathrm{g})_{11}^{1/2}$, $r_2 = (\mathrm{g})_{12}^{1/2}$, $r_3 = (\mathrm{g})_{13}^{1/2}$, $r_4 = (\mathrm{g})_{14}^{1/2}$, $r_5 = (\mathrm{g})_{22}^{1/2}$, and $r_6 = (\mathrm{g})_{23}^{1/2}$. The conditions (60), (61), and (62) form a system of equations for unknowns A, B, C, and D. Unfortunately, it is still difficult to solve this system of equations analytically. However, it has been verified that the same results as the examples shown in Section II.*D* are numerically obtained from the system of equations. Although it is just only a numerical verification, we expect the set of equations mentioned above correctly captures an algebraic structure of the optimal detection. .

In the case of minimax detection, the following conditions are enforced:

- Symmetry of the minimax distribution.

$$p_1^\circ = p_4^\circ \quad \text{and} \quad p_2^\circ = p_3^\circ, \tag{63}$$

  and $p_1^\circ + p_2^\circ + p_3^\circ + p_4^\circ = 1$ and $p_i^\circ \geq 0$ for every i.
- Bayes-optimality at the minimax distribution.

$$\begin{aligned}
&p_1^\circ (r_1 A + r_2 B + r_3 C + r_4 D) \\
&\quad \times (r_2 A - r_1 B - r_4 C + r_3 D) \\
&\qquad = p_2^\circ (r_2 A + r_5 B + r_6 C + r_3 D) \\
&\qquad\quad \times (r_5 A - r_2 B - r_3 C + r_6 D); \qquad (64) \\
&p_1^\circ (r_1 A + r_2 B + r_3 C + r_4 D) \\
&\quad \times (r_3 A - r_4 B - r_1 C + r_2 D) \\
&\qquad = p_2^\circ (r_3 A + r_6 B + r_5 C + r_2 D) \\
&\qquad\quad \times (r_5 A - r_2 B - r_3 C + r_6 D). \qquad (65)
\end{aligned}$$

- Minimax condition.

$$\begin{aligned}
&r_1 A + r_2 B + r_3 C + r_4 D \\
&\quad = r_5 A - r_2 B - r_3 C + r_6 D. \qquad (66)
\end{aligned}$$

We expect the conditions, (60) and (63)-(66), will be helpful for finding the analytical solution to the minimax detection problem of QASK.

IV. Summary

Quantum detection of quaternary amplitude-shift keying (QASK) coherent state signal was investigated. The closed-form expression of the square-root measurement (SRM) for QASK coherent state signal was derived by solving the eigenvalue problem of the corresponding Gram matrix. The optimal detection vectors of the Bayes-optimal detection at the uniform distribution of signal elements and the minimax detection were respectively calculated by Nakahira's iterative calculation algorithms [15]. Toward derivation of the closed-form expressions of the Bayes-optimal detection and the minimax detection, the structure of optimal detection vectors was discussed based on the calculation results. From this, templates of the optimal detection vectors for the Bayes-optimal detection at the uniform signal distribution and for the minimax detection were conjectured. Based on this conjecture, the systems of equations to determine the corresponding optimal detection vectors for QASK coherent state signal were proposed. Thus the derivation of the closed-form expressions for the Bayes-optimal detection at the uniform signal distribution and for the minimax detection for QASK coherent state signal are still remaining. This problem will be discussed else where.

Acknowledgment

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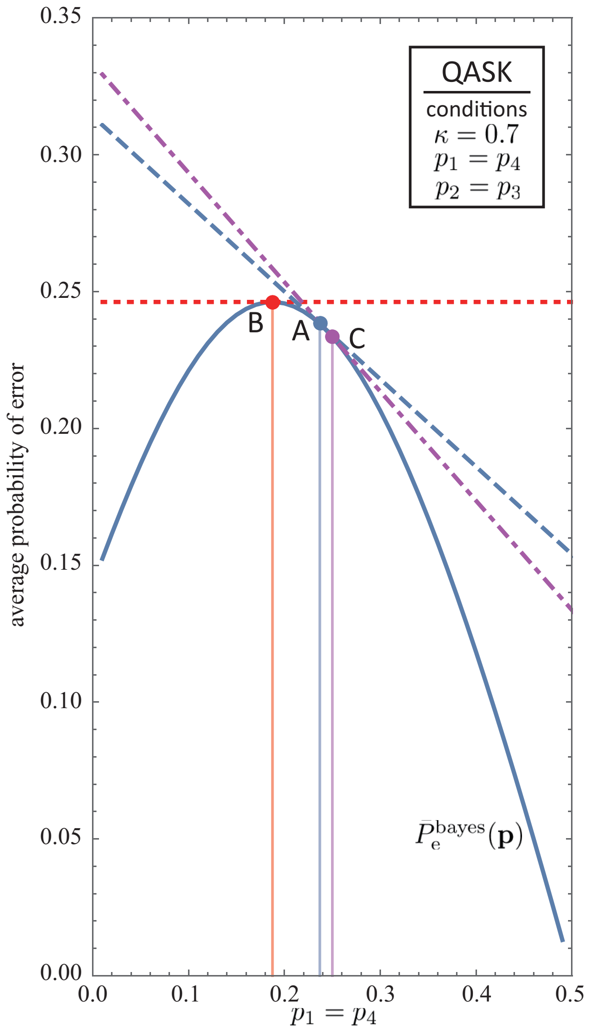

Fig. 6. SRM, minimax, and Bayes-optimal cases for $\kappa = 0.7$

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Appendix

A. Definition of QASK

We have defined QASK coherent state signal by Eq.(1). However, QASK coherent state signal may be defined as follows.

$$\mathcal{S}' = \{|0\rangle, |2\alpha\rangle, |4\alpha\rangle, |6\alpha\rangle\}.$$

When this type of definition is employed, the average number of signal photons is given as $\bar{n}_{\mathrm{s}} = 14|\alpha|^2$. Even in this case, the Gram matrix is given by Eq.(12).

B. Numerical example of the square-root of Gram matrix

The following examples are obtained from Eqs.(13)-(16) and (17)-(20).

<u>Case of $\kappa = 0.1$</u>
Gram matrix G:

$$\begin{bmatrix} \texttt{1.0000E+0} & \texttt{1.0000E-1} & \texttt{1.0000E-4} & \texttt{1.0000E-9} \\ \texttt{1.0000E-1} & \texttt{1.0000E+0} & \texttt{1.0000E-1} & \texttt{1.0000E-4} \\ \texttt{1.0000E-4} & \texttt{1.0000E-1} & \texttt{1.0000E+0} & \texttt{1.0000E-1} \\ \texttt{1.0000E-9} & \texttt{1.0000E-4} & \texttt{1.0000E-1} & \texttt{1.0000E+0} \end{bmatrix}.$$

Eigenvalues and eigenvectors of G:

$$\lambda_1 = 0.83829, \qquad \vec{\lambda}_1 = \begin{bmatrix} 0.37163 \\ -0.60158 \\ 0.60158 \\ -0.37163 \end{bmatrix};$$

$$\lambda_2 = 0.93811, \qquad \vec{\lambda}_2 = \begin{bmatrix} 0.60143 \\ -0.37187 \\ -0.37187 \\ 0.60143 \end{bmatrix};$$

$$\lambda_3 = 1.0617, \qquad \vec{\lambda}_3 = \begin{bmatrix} 0.60158 \\ 0.37163 \\ -0.37163 \\ -0.60158 \end{bmatrix};$$

$$\lambda_4 = 1.1619, \qquad \vec{\lambda}_4 = \begin{bmatrix} 0.37187 \\ 0.60143 \\ 0.60143 \\ 0.37187 \end{bmatrix}.$$

Square-root of G, $G^{1/2}$:

$$\begin{bmatrix} 9.9874E-1 & 5.0125E-2 & -1.2116E-3 & 6.0810E-5 \\ 5.0125E-2 & 9.9748E-1 & 5.0187E-2 & -1.2116E-3 \\ -1.2116E-3 & 5.0187E-2 & 9.9748E-1 & 5.0125E-2 \\ 6.0810E-5 & -1.2116E-3 & 5.0125E-2 & 9.9874E-1 \end{bmatrix}.$$

case of $\kappa = 0.3$

Gram matrix G:

$$\begin{bmatrix} 1.0000E+0 & 3.0000E-1 & 8.1000E-3 & 1.9683E-5 \\ 3.0000E-1 & 1.0000E+0 & 3.0000E-1 & 8.1000E-3 \\ 8.1000E-3 & 3.0000E-1 & 1.0000E+0 & 3.0000E-1 \\ 1.9683E-5 & 8.1000E-3 & 3.0000E-1 & 1.0000E+0 \end{bmatrix}.$$

Eigenvalues and eigenvectors of G:

$$\lambda_1 = 0.52181, \qquad \vec{\lambda}_1 = \begin{bmatrix} 0.36843 \\ -0.60354 \\ 0.60354 \\ -0.36843 \end{bmatrix};$$

$$\lambda_2 = 0.80734, \qquad \vec{\lambda}_2 = \begin{bmatrix} 0.59952 \\ -0.37493 \\ -0.37493 \\ 0.59952 \end{bmatrix};$$

$$\lambda_3 = 1.1782, \qquad \vec{\lambda}_3 = \begin{bmatrix} 0.60354 \\ 0.36843 \\ -0.36843 \\ -0.60354 \end{bmatrix};$$

$$\lambda_4 = 1.4927, \qquad \vec{\lambda}_4 = \begin{bmatrix} 0.37493 \\ 0.59952 \\ 0.59952 \\ 0.37493 \end{bmatrix}.$$

Square-root of G, $G^{1/2}$:

$$\begin{bmatrix} 9.8813E-1 & 1.5339E-1 & -8.0778E-3 & 1.2639E-3 \\ 1.5339E-1 & 9.7591E-1 & 1.5497E-1 & -8.0778E-3 \\ -8.0778E-3 & 1.5497E-1 & 9.7591E-1 & 1.5339E-1 \\ 1.2639E-3 & -8.0778E-3 & 1.5339E-1 & 9.8813E-1 \end{bmatrix}.$$

case of $\kappa = 0.5$

Gram matrix G:

$$\begin{bmatrix} 1.0000E+0 & 5.0000E-1 & 6.2500E-2 & 1.9531E-3 \\ 5.0000E-1 & 1.0000E+0 & 5.0000E-1 & 6.2500E-2 \\ 6.2500E-2 & 5.0000E-1 & 1.0000E+0 & 5.0000E-1 \\ 1.9531E-3 & 6.2500E-2 & 5.0000E-1 & 1.0000E+0 \end{bmatrix}.$$

Eigenvalues and eigenvectors of G:

$$\lambda_1 = 0.24562, \qquad \vec{\lambda}_1 = \begin{bmatrix} 0.35543 \\ -0.61129 \\ 0.61129 \\ -0.35543 \end{bmatrix};$$

$$\lambda_2 = 0.63582, \qquad \vec{\lambda}_2 = \begin{bmatrix} 0.59262 \\ -0.38574 \\ -0.38574 \\ 0.59262 \end{bmatrix};$$

$$\lambda_3 = 1.2524, \qquad \vec{\lambda}_3 = \begin{bmatrix} 0.61129 \\ 0.35543 \\ -0.35543 \\ -0.61129 \end{bmatrix};$$

$$\lambda_4 = 1.8661, \qquad \vec{\lambda}_4 = \begin{bmatrix} 0.38574 \\ 0.59262 \\ 0.59262 \\ 0.38574 \end{bmatrix}.$$

Square-root of G, $G^{1/2}$:

$$\begin{bmatrix} 9.6410E-1 & 2.6547E-1 & -5.4718E-3 & 2.5196E-3 \\ 2.6547E-1 & 9.2498E-1 & 2.7185E-1 & -5.4718E-3 \\ -5.4718E-3 & 2.7185E-1 & 9.2498E-1 & 2.6547E-1 \\ 2.5196E-3 & -5.4718E-3 & 2.6547E-1 & 9.6410E-1 \end{bmatrix}.$$

case of $\kappa = 0.7$

Gram matrix G:

$$\begin{bmatrix} 1.0000E+0 & 7.0000E-1 & 2.4010E-1 & 4.0354E-2 \\ 7.0000E-1 & 1.0000E+0 & 7.0000E-1 & 2.4010E-1 \\ 2.4010E-1 & 7.0000E-1 & 1.0000E+0 & 7.0000E-1 \\ 4.0354E-2 & 2.4010E-1 & 7.0000E-1 & 1.0000E+0 \end{bmatrix}.$$

Eigenvalues and eigenvectors of G:

$$\lambda_1 = 0.063880, \qquad \vec{\lambda}_1 = \begin{bmatrix} 0.32296 \\ -0.62904 \\ 0.62904 \\ -0.32296 \end{bmatrix};$$

$$\lambda_2 = 0.37390, \qquad \vec{\lambda}_2 = \begin{bmatrix} 0.57686 \\ -0.40895 \\ -0.40895 \\ 0.57686 \end{bmatrix};$$

$$\lambda_3 = 1.1958, \qquad \vec{\lambda}_3 = \begin{bmatrix} 0.62904 \\ 0.32296 \\ -0.32296 \\ -0.62904 \end{bmatrix};$$

$$\lambda_4 = 2.3665, \qquad \vec{\lambda}_4 = \begin{bmatrix} 0.40895 \\ 0.57686 \\ 0.57686 \\ 0.40895 \end{bmatrix}.$$

Square-root of G, $G^{1/2}$:

$$\begin{bmatrix} 9.1980E-1 & 3.8946E-1 & 4.7841E-2 & 1.6795E-3 \\ 3.8946E-1 & 8.2823E-1 & 4.0009E-1 & 4.7841E-2 \\ 4.7841E-2 & 4.0009E-1 & 8.2823E-1 & 3.8946E-1 \\ 1.6795E-3 & 4.7841E-2 & 3.8946E-1 & 9.1980E-1 \end{bmatrix}.$$

<u>**case of** $\kappa = 0.9$</u>
Gram matrix G:

$$\begin{bmatrix} 1.0000E+0 & 9.0000E-1 & 6.5610E-1 & 3.8742E-1 \\ 9.0000E-1 & 1.0000E+0 & 9.0000E-1 & 6.5610E-1 \\ 6.5610E-1 & 9.0000E-1 & 1.0000E+0 & 9.0000E-1 \\ 3.8742E-1 & 6.5610E-1 & 9.0000E-1 & 1.0000E+0 \end{bmatrix}.$$

Eigenvalues and eigenvectors of G:

$$\lambda_1 = 0.002494, \qquad \vec{\lambda}_1 = \begin{bmatrix} 0.26249 \\ -0.65658 \\ 0.65658 \\ -0.26249 \end{bmatrix};$$

$$\lambda_2 = 0.06665, \qquad \vec{\lambda}_2 = \begin{bmatrix} 0.53910 \\ -0.45757 \\ -0.45757 \\ 0.53910 \end{bmatrix};$$

$$\lambda_3 = 0.71009, \qquad \vec{\lambda}_3 = \begin{bmatrix} 0.65658 \\ 0.26249 \\ -0.26249 \\ -0.65658 \end{bmatrix};$$

$$\lambda_4 = 3.2208, \qquad \vec{\lambda}_4 = \begin{bmatrix} 0.45757 \\ 0.53910 \\ 0.53910 \\ 0.45757 \end{bmatrix}.$$

Square-root of G, $G^{1/2}$:

$$\begin{bmatrix} 8.1749E-1 & 5.1564E-1 & 2.4239E-1 & 8.4065E-2 \\ 5.1564E-1 & 6.5521E-1 & 4.9604E-1 & 2.4239E-1 \\ 2.4239E-1 & 4.9604E-1 & 6.5521E-1 & 5.1564E-1 \\ 8.4065E-2 & 2.4239E-1 & 5.1564E-1 & 8.1749E-1 \end{bmatrix}.$$

C. Numerical example of the column vector representation of the signal elements

<u>**case of** $\kappa = 0.1$</u>

$$|\psi_1\rangle \doteq [|\psi_1\rangle]_\gamma = \begin{bmatrix} 9.9874E-1 \\ 5.0125E-2 \\ -1.2116E-3 \\ 6.0810E-5 \end{bmatrix};$$

$$|\psi_2\rangle \doteq [|\psi_2\rangle]_\gamma = \begin{bmatrix} 5.0125E-2 \\ 9.9748E-1 \\ 5.0187E-2 \\ -1.2116E-3 \end{bmatrix};$$

$$|\psi_3\rangle \doteq [|\psi_3\rangle]_\gamma = \begin{bmatrix} -1.2116E-3 \\ 5.0187E-2 \\ 9.9748E-1 \\ 5.0125E-2 \end{bmatrix};$$

$$|\psi_4\rangle \doteq [|\psi_4\rangle]_\gamma = \begin{bmatrix} 6.0810E-5 \\ -1.2116E-3 \\ 5.0125E-2 \\ 9.9874E-1 \end{bmatrix}.$$

<u>**case of** $\kappa = 0.3$</u>

$$|\psi_1\rangle \doteq [|\psi_1\rangle]_\gamma = \begin{bmatrix} 9.8813E-1 \\ 1.5339E-1 \\ -8.0778E-3 \\ 1.2639E-3 \end{bmatrix};$$

$$|\psi_2\rangle \doteq [|\psi_2\rangle]_\gamma = \begin{bmatrix} 1.5339E-1 \\ 9.7591E-1 \\ 1.5497E-1 \\ -8.0778E-3 \end{bmatrix};$$

$$|\psi_3\rangle \doteq [|\psi_3\rangle]_\gamma = \begin{bmatrix} -8.0778E-3 \\ 1.5497E-1 \\ 9.7591E-1 \\ 1.5339E-1 \end{bmatrix};$$

$$|\psi_4\rangle \doteq [|\psi_4\rangle]_\gamma = \begin{bmatrix} 1.2639E-3 \\ -8.0778E-3 \\ 1.5339E-1 \\ 9.8813E-1 \end{bmatrix}.$$

<u>**case of** $\kappa = 0.5$</u>

$$|\psi_1\rangle \doteq [|\psi_1\rangle]_\gamma = \begin{bmatrix} 9.6410E-1 \\ 2.6547E-1 \\ -5.4718E-3 \\ 2.5196E-3 \end{bmatrix};$$

$$|\psi_2\rangle \doteq [|\psi_2\rangle]_\gamma = \begin{bmatrix} 2.6547E-1 \\ 9.2498E-1 \\ 2.7185E-1 \\ -5.4718E-3 \end{bmatrix};$$

$$|\psi_3\rangle \doteq [|\psi_3\rangle]_\gamma = \begin{bmatrix} -5.4718E-3 \\ 2.7185E-1 \\ 9.2498E-1 \\ 2.6547E-1 \end{bmatrix};$$

$$|\psi_4\rangle \doteq [|\psi_4\rangle]_\gamma = \begin{bmatrix} 2.5196E-3 \\ -5.4718E-3 \\ 2.6547E-1 \\ 9.6410E-1 \end{bmatrix}.$$

<u>**case of** $\kappa = 0.7$</u>

$$|\psi_1\rangle \doteq [|\psi_1\rangle]_\gamma = \begin{bmatrix} 9.1980E-1 \\ 3.8946E-1 \\ 4.7841E-2 \\ 1.6795E-3 \end{bmatrix};$$

$$|\psi_2\rangle \doteq [|\psi_2\rangle]_\gamma = \begin{bmatrix} 3.8946E-1 \\ 8.2823E-1 \\ 4.0009E-1 \\ 4.7841E-2 \end{bmatrix};$$

$$|\psi_3\rangle \doteq [|\psi_3\rangle]_\gamma = \begin{bmatrix} 4.7841E-2 \\ 4.0009E-1 \\ 8.2823E-1 \\ 3.8946E-1 \end{bmatrix};$$

$$|\psi_4\rangle \doteq [|\psi_4\rangle]_\gamma = \begin{bmatrix} \texttt{1.6795E-3} \\ \texttt{4.7841E-2} \\ \texttt{3.8946E-1} \\ \texttt{9.1980E-1} \end{bmatrix}.$$

case of $\kappa = 0.9$

$$|\psi_1\rangle \doteq [|\psi_1\rangle]_\gamma = \begin{bmatrix} \texttt{8.1749E-1} \\ \texttt{5.1564E-1} \\ \texttt{2.4239E-1} \\ \texttt{8.4065E-2} \end{bmatrix};$$

$$|\psi_2\rangle \doteq [|\psi_2\rangle]_\gamma = \begin{bmatrix} \texttt{5.1564E-1} \\ \texttt{6.5521E-1} \\ \texttt{4.9604E-1} \\ \texttt{2.4239E-1} \end{bmatrix};$$

$$|\psi_3\rangle \doteq [|\psi_3\rangle]_\gamma = \begin{bmatrix} \texttt{2.4239E-1} \\ \texttt{4.9604E-1} \\ \texttt{6.5521E-1} \\ \texttt{5.1564E-1} \end{bmatrix};$$

$$|\psi_4\rangle \doteq [|\psi_4\rangle]_\gamma = \begin{bmatrix} \texttt{8.4065E-2} \\ \texttt{2.4239E-1} \\ \texttt{5.1564E-1} \\ \texttt{8.1749E-1} \end{bmatrix}.$$

D. Numerical example of the SRM

case of $\kappa = 0.1$
Optimal distribution $\mathbf{p}^\bullet$ of signal:

$$p_1^\bullet = p_4^\bullet = \texttt{0.24984};$$

$$p_2^\bullet = p_3^\bullet = \texttt{0.25016}.$$

Channel matrix $[P(j|i)] = [\langle\psi_i|\hat{\Pi}_j^\bullet|\psi_i\rangle]$:

$$\begin{bmatrix} \texttt{9.9749E-1} & \texttt{2.5125E-3} & \texttt{1.4681E-6} & \texttt{3.6979E-9} \\ \texttt{2.5125E-3} & \texttt{9.9497E-1} & \texttt{2.5188E-3} & \texttt{1.4681E-6} \\ \texttt{1.4681E-6} & \texttt{2.5188E-3} & \texttt{9.9497E-1} & \texttt{2.5125E-3} \\ \texttt{3.6979E-9} & \texttt{1.4681E-6} & \texttt{2.5125E-3} & \texttt{9.9749E-1} \end{bmatrix}.$$

Minimal average probability of error, $\bar{P}_e^\bullet$:

$$\bar{P}_e^\bullet = \texttt{3.7742E-3}.$$

case of $\kappa = 0.3$
Optimal distribution $\mathbf{p}^\bullet$ of signal:

$$p_1^\bullet = p_4^\bullet = \texttt{0.24844};$$

$$p_2^\bullet = p_3^\bullet = \texttt{0.25156}.$$

Channel matrix $[P(j|i)] = [\langle\psi_i|\hat{\Pi}_j^\bullet|\psi_i\rangle]$:

$$\begin{bmatrix} \texttt{9.7641E-1} & \texttt{2.3528E-2} & \texttt{6.5251E-5} & \texttt{1.5974E-6} \\ \texttt{2.3528E-2} & \texttt{9.5239E-1} & \texttt{2.4017E-2} & \texttt{6.5251E-5} \\ \texttt{6.5251E-5} & \texttt{2.4017E-2} & \texttt{9.5239E-1} & \texttt{2.3528E-2} \\ \texttt{1.5974E-6} & \texttt{6.5251E-5} & \texttt{2.3528E-2} & \texttt{9.7641E-1} \end{bmatrix}.$$

Minimal average probability of error, $\bar{P}_e^\bullet$:

$$\bar{P}_e^\bullet = \texttt{3.5677E-2}.$$

case of $\kappa = 0.5$
Optimal distribution $\mathbf{p}^\bullet$ of signal:

$$p_1^\bullet = p_4^\bullet = \texttt{0.24482};$$

$$p_2^\bullet = p_3^\bullet = \texttt{0.25518}.$$

Channel matrix $[P(j|i)] = [\langle\psi_i|\hat{\Pi}_j^\bullet|\psi_i\rangle]$:

$$\begin{bmatrix} \texttt{9.2949E-1} & \texttt{7.0476E-2} & \texttt{2.9941E-5} & \texttt{6.3486E-6} \\ \texttt{7.0476E-2} & \texttt{8.5559E-1} & \texttt{7.3900E-2} & \texttt{2.9941E-5} \\ \texttt{2.9941E-5} & \texttt{7.3900E-2} & \texttt{8.5559E-1} & \texttt{7.0476E-2} \\ \texttt{6.3486E-6} & \texttt{2.9941E-5} & \texttt{7.0476E-2} & \texttt{9.2949E-1} \end{bmatrix}.$$

Minimal average probability of error, $\bar{P}_e^\bullet$:

$$\bar{P}_e^\bullet = \texttt{0.10822}.$$

case of $\kappa = 0.7$
Optimal distribution $\mathbf{p}^\bullet$ of signal:

$$p_1^\bullet = p_4^\bullet = \texttt{0.23690};$$

$$p_2^\bullet = p_3^\bullet = \texttt{0.26310}.$$

Channel matrix $[P(j|i)] = [\langle\psi_i|\hat{\Pi}_j^\bullet|\psi_i\rangle]$:

$$\begin{bmatrix} \texttt{8.4603E-1} & \texttt{1.5168E-1} & \texttt{2.2888E-3} & \texttt{2.8207E-6} \\ \texttt{1.5168E-1} & \texttt{6.8596E-1} & \texttt{1.6007E-1} & \texttt{2.2888E-3} \\ \texttt{2.2888E-3} & \texttt{1.6007E-1} & \texttt{6.8596E-1} & \texttt{1.5168E-1} \\ \texttt{2.8207E-6} & \texttt{2.2888E-3} & \texttt{1.5168E-1} & \texttt{8.4603E-1} \end{bmatrix}.$$

Minimal average probability of error, $\bar{P}_e^\bullet$:

$$\bar{P}_e^\bullet = \texttt{0.23820}.$$

case of $\kappa = 0.9$
Optimal distribution $\mathbf{p}^\bullet$ of signal:

$$p_1^\bullet = p_4^\bullet = \texttt{0.22245};$$

$$p_2^\bullet = p_3^\bullet = \texttt{0.27755}.$$

Channel matrix $[P(j|i)] = [\langle\psi_i|\hat{\Pi}_j^\bullet|\psi_i\rangle]$:

$$\begin{bmatrix} \texttt{6.6829E-1} & \texttt{2.6588E-1} & \texttt{5.8755E-2} & \texttt{7.0669E-3} \\ \texttt{2.6588E-1} & \texttt{4.2931E-1} & \texttt{2.4606E-1} & \texttt{5.8755E-2} \\ \texttt{5.8755E-2} & \texttt{2.4606E-1} & \texttt{4.2931E-1} & \texttt{2.6588E-1} \\ \texttt{7.0669E-3} & \texttt{5.8755E-2} & \texttt{2.6588E-1} & \texttt{6.6829E-1} \end{bmatrix}.$$

Minimal average probability of error, $\bar{P}_e^\bullet$:

$$\bar{P}_e^\bullet = \texttt{0.46437}.$$

E. Numerical example of the Bayes-optimal detection at the uniform signal distribution

case of $\kappa = 0.1$
Detection vectors:

$$|\mu_1^{\text{bayes}}(\mathbf{u})\rangle \doteq [|\mu_1^{\text{bayes}}(\mathbf{u})\rangle]_\gamma = \begin{bmatrix} \texttt{1.0000E+0} \\ \texttt{3.1863E-5} \\ \texttt{-1.5708E-6} \\ \texttt{5.0048E-11} \end{bmatrix};$$

$$|\mu_2^{\text{bayes}}(\mathbf{u})\rangle \doteq [|\mu_2^{\text{bayes}}(\mathbf{u})\rangle]_\gamma = \begin{bmatrix} \texttt{-3.1863E-5} \\ \texttt{1.0000E+0} \\ \texttt{5.0048E-11} \\ \texttt{1.5708E-6} \end{bmatrix};$$

$$|\mu_3^{\text{bayes}}(\mathbf{u})\rangle \doteq [|\mu_3^{\text{bayes}}(\mathbf{u})\rangle]_\gamma = \begin{bmatrix} 1.5708\text{E}{-6} \\ 5.0048\text{E}{-11} \\ 1.0000\text{E}{+0} \\ -3.1863\text{E}{-5} \end{bmatrix};$$

$$|\mu_4^{\text{bayes}}(\mathbf{u})\rangle \doteq [|\mu_4^{\text{bayes}}(\mathbf{u})\rangle]_\gamma = \begin{bmatrix} 5.0048\text{E}{-11} \\ -1.5708\text{E}{-6} \\ 3.1863\text{E}{-5} \\ 1.0000\text{E}{+0} \end{bmatrix}.$$

Note that 1.0000E+0 in this example is not 1 with machine accuracy. It is very close to 1, but below 1.
Channel matrix $[P(j|i)] = [\langle\psi_i|\hat{\Pi}_j^{\text{bayes}}(\mathbf{u})|\psi_i\rangle]$:

$$\begin{bmatrix} 9.9749\text{E}{-1} & 2.5093\text{E}{-3} & 1.4643\text{E}{-6} & 3.6836\text{E}{-9} \\ 2.5157\text{E}{-3} & 9.9496\text{E}{-1} & 2.5188\text{E}{-3} & 1.4680\text{E}{-6} \\ 1.4680\text{E}{-6} & 2.5188\text{E}{-3} & 9.9496\text{E}{-1} & 2.5157\text{E}{-3} \\ 3.6836\text{E}{-9} & 1.4643\text{E}{-6} & 2.5093\text{E}{-3} & 9.9749\text{E}{-1} \end{bmatrix}.$$

Minimal average probability of error, $\bar{P}_{\text{e}}^{\text{bayes}}(\mathbf{u})$:

$$\bar{P}_{\text{e}}^{\text{bayes}}(\mathbf{u}) = 3.7734\text{E}{-3}.$$

case of $\kappa = 0.3$
Detection vectors:

$$|\mu_1^{\text{bayes}}(\mathbf{u})\rangle \doteq [|\mu_1^{\text{bayes}}(\mathbf{u})\rangle]_\gamma = \begin{bmatrix} 1.0000\text{E}{+0} \\ 1.0076\text{E}{-3} \\ -1.3216\text{E}{-4} \\ 1.3316\text{E}{-7} \end{bmatrix};$$

$$|\mu_2^{\text{bayes}}(\mathbf{u})\rangle \doteq [|\mu_2^{\text{bayes}}(\mathbf{u})\rangle]_\gamma = \begin{bmatrix} -1.0076\text{E}{-3} \\ 1.0000\text{E}{+0} \\ 1.3316\text{E}{-7} \\ 1.3216\text{E}{-4} \end{bmatrix};$$

$$|\mu_3^{\text{bayes}}(\mathbf{u})\rangle \doteq [|\mu_3^{\text{bayes}}(\mathbf{u})\rangle]_\gamma = \begin{bmatrix} 1.3216\text{E}{-4} \\ 1.3316\text{E}{-7} \\ 1.0000\text{E}{+0} \\ -1.0076\text{E}{-3} \end{bmatrix};$$

$$|\mu_4^{\text{bayes}}(\mathbf{u})\rangle \doteq [|\mu_4^{\text{bayes}}(\mathbf{u})\rangle]_\gamma = \begin{bmatrix} 1.3316\text{E}{-7} \\ -1.3216\text{E}{-4} \\ 1.0076\text{E}{-3} \\ 1.0000\text{E}{+0} \end{bmatrix}.$$

Channel matrix $[P(j|i)] = [\langle\psi_i|\hat{\Pi}_j^{\text{bayes}}(\mathbf{u})|\psi_i\rangle]$:

$$\begin{bmatrix} 9.7671\text{E}{-1} & 2.3224\text{E}{-2} & 6.3178\text{E}{-5} & 1.5267\text{E}{-6} \\ 2.3825\text{E}{-2} & 9.5209\text{E}{-1} & 2.4026\text{E}{-2} & 6.4812\text{E}{-5} \\ 6.4812\text{E}{-5} & 2.4026\text{E}{-2} & 9.5209\text{E}{-1} & 2.3825\text{E}{-2} \\ 1.5267\text{E}{-6} & 6.3178\text{E}{-5} & 2.3224\text{E}{-2} & 9.7671\text{E}{-1} \end{bmatrix}.$$

Minimal average probability of error, $\bar{P}_{\text{e}}^{\text{bayes}}(\mathbf{u})$:

$$\bar{P}_{\text{e}}^{\text{bayes}}(\mathbf{u}) = 3.5602\text{E}{-2}.$$

case of $\kappa = 0.5$
Detection vectors:

$$|\mu_1^{\text{bayes}}(\mathbf{u})\rangle \doteq [|\mu_1^{\text{bayes}}(\mathbf{u})\rangle]_\gamma = \begin{bmatrix} 9.9998\text{E}{-1} \\ 6.4790\text{E}{-3} \\ -1.0502\text{E}{-3} \\ 6.8043\text{E}{-6} \end{bmatrix};$$

$$|\mu_2^{\text{bayes}}(\mathbf{u})\rangle \doteq [|\mu_2^{\text{bayes}}(\mathbf{u})\rangle]_\gamma = \begin{bmatrix} -6.4790\text{E}{-3} \\ 9.9998\text{E}{-1} \\ 6.8043\text{E}{-6} \\ 1.0502\text{E}{-3} \end{bmatrix};$$

$$|\mu_3^{\text{bayes}}(\mathbf{u})\rangle \doteq [|\mu_3^{\text{bayes}}(\mathbf{u})\rangle]_\gamma = \begin{bmatrix} 1.0502\text{E}{-3} \\ 6.8043\text{E}{-6} \\ 9.9998\text{E}{-1} \\ -6.4790\text{E}{-3} \end{bmatrix};$$

$$|\mu_4^{\text{bayes}}(\mathbf{u})\rangle \doteq [|\mu_4^{\text{bayes}}(\mathbf{u})\rangle]_\gamma = \begin{bmatrix} 6.8043\text{E}{-6} \\ -1.0502\text{E}{-3} \\ 6.4790\text{E}{-3} \\ 9.9998\text{E}{-1} \end{bmatrix}.$$

Channel matrix $[P(j|i)] = [\langle\psi_i|\hat{\Pi}_j^{\text{bayes}}(\mathbf{u})|\psi_i\rangle]$:

$$\begin{bmatrix} 9.3278\text{E}{-1} & 6.7197\text{E}{-2} & 2.0014\text{E}{-5} & 4.8925\text{E}{-6} \\ 7.3536\text{E}{-2} & 8.5237\text{E}{-1} & 7.4071\text{E}{-2} & 2.1902\text{E}{-5} \\ 2.1902\text{E}{-5} & 7.4071\text{E}{-2} & 8.5237\text{E}{-1} & 7.3536\text{E}{-2} \\ 4.8925\text{E}{-6} & 2.0014\text{E}{-5} & 6.7197\text{E}{-2} & 9.3278\text{E}{-1} \end{bmatrix}.$$

Minimal average probability of error, $\bar{P}_{\text{e}}^{\text{bayes}}(\mathbf{u})$:

$$\bar{P}_{\text{e}}^{\text{bayes}}(\mathbf{u}) = 0.10743.$$

case of $\kappa = 0.7$
Detection vectors:

$$|\mu_1^{\text{bayes}}(\mathbf{u})\rangle \doteq [|\mu_1^{\text{bayes}}(\mathbf{u})\rangle]_\gamma = \begin{bmatrix} 9.9956\text{E}{-1} \\ 2.9636\text{E}{-2} \\ -2.7782\text{E}{-3} \\ 8.2371\text{E}{-5} \end{bmatrix};$$

$$|\mu_2^{\text{bayes}}(\mathbf{u})\rangle \doteq [|\mu_2^{\text{bayes}}(\mathbf{u})\rangle]_\gamma = \begin{bmatrix} -2.9636\text{E}{-2} \\ 9.9956\text{E}{-1} \\ 8.2371\text{E}{-5} \\ 2.7782\text{E}{-3} \end{bmatrix};$$

$$|\mu_3^{\text{bayes}}(\mathbf{u})\rangle \doteq [|\mu_3^{\text{bayes}}(\mathbf{u})\rangle]_\gamma = \begin{bmatrix} 2.7782\text{E}{-3} \\ 8.2371\text{E}{-5} \\ 9.9956\text{E}{-1} \\ -2.9636\text{E}{-2} \end{bmatrix};$$

$$|\mu_4^{\text{bayes}}(\mathbf{u})\rangle \doteq [|\mu_4^{\text{bayes}}(\mathbf{u})\rangle]_\gamma = \begin{bmatrix} 8.2371\text{E}{-5} \\ -2.7782\text{E}{-3} \\ 2.9636\text{E}{-2} \\ 9.9956\text{E}{-1} \end{bmatrix}.$$

Channel matrix $[P(j|i)] = [\langle\psi_i|\hat{\Pi}_j^{\text{bayes}}(\mathbf{u})|\psi_i\rangle]$:

$$\begin{bmatrix} 8.6639\text{E}{-1} & 1.3107\text{E}{-1} & 2.5359\text{E}{-3} & 4.3697\text{E}{-6} \\ 1.7034\text{E}{-1} & 6.6665\text{E}{-1} & 1.5972\text{E}{-1} & 3.2957\text{E}{-3} \\ 3.2957\text{E}{-3} & 1.5972\text{E}{-1} & 6.6665\text{E}{-1} & 1.7034\text{E}{-1} \\ 4.3697\text{E}{-6} & 2.5359\text{E}{-3} & 1.3107\text{E}{-1} & 8.6639\text{E}{-1} \end{bmatrix}.$$

Minimal average probability of error, $\bar{P}_{\text{e}}^{\text{bayes}}(\mathbf{u})$:

$$\bar{P}_{\text{e}}^{\text{bayes}}(\mathbf{u}) = 0.23348.$$

case of $\kappa = 0.9$

Detection vectors:

$$|\mu_1^{\rm bayes}(\mathbf{u})\rangle \doteq [|\mu_1^{\rm bayes}(\mathbf{u})\rangle]_\gamma = \begin{bmatrix} 9.8990\text{E}{-1} \\ 1.3977\text{E}{-1} \\ 2.3468\text{E}{-2} \\ -3.3136\text{E}{-3} \end{bmatrix};$$

$$|\mu_2^{\rm bayes}(\mathbf{u})\rangle \doteq [|\mu_2^{\rm bayes}(\mathbf{u})\rangle]_\gamma = \begin{bmatrix} -1.3977\text{E}{-1} \\ 9.8990\text{E}{-1} \\ -3.3136\text{E}{-3} \\ -2.3468\text{E}{-2} \end{bmatrix};$$

$$|\mu_3^{\rm bayes}(\mathbf{u})\rangle \doteq [|\mu_3^{\rm bayes}(\mathbf{u})\rangle]_\gamma = \begin{bmatrix} -2.3468\text{E}{-2} \\ -3.3136\text{E}{-3} \\ 9.8990\text{E}{-1} \\ -1.3977\text{E}{-1} \end{bmatrix};$$

$$|\mu_4^{\rm bayes}(\mathbf{u})\rangle \doteq [|\mu_4^{\rm bayes}(\mathbf{u})\rangle]_\gamma = \begin{bmatrix} -3.3136\text{E}{-3} \\ 2.3468\text{E}{-2} \\ 1.3977\text{E}{-1} \\ 9.8990\text{E}{-1} \end{bmatrix},$$

Channel matrix $[P(j|i)] = [\langle\psi_i|\hat{\Pi}_j^{\rm bayes}(\mathbf{u})|\psi_i\rangle]$:

$$\begin{bmatrix} 7.8627\text{E}{-1} & 1.5476\text{E}{-1} & 4.2974\text{E}{-2} & 1.5999\text{E}{-2} \\ 3.7558\text{E}{-1} & 3.2398\text{E}{-1} & 1.9614\text{E}{-1} & 1.0429\text{E}{-1} \\ 1.0429\text{E}{-1} & 1.9614\text{E}{-1} & 3.2398\text{E}{-1} & 3.7558\text{E}{-1} \\ 1.5999\text{E}{-2} & 4.2974\text{E}{-2} & 1.5476\text{E}{-1} & 7.8627\text{E}{-1} \end{bmatrix}.$$

Minimal average probability of error, $\bar{P}_{\rm e}^{\rm bayes}(\mathbf{u})$:

$$\bar{P}_{\rm e}^{\rm bayes}(\mathbf{u}) = 0.44488.$$

F. Numerical example of the minimax detection

<u>**case of** $\kappa = 0.1$</u>
Optimal distribution $\mathbf{p}^\circ$:

$$p_1^\circ = 0.18757;$$
$$p_2^\circ = 0.31243;$$
$$p_3^\circ = 0.31243;$$
$$p_4^\circ = 0.18757.$$

Detection vectors:

$$|\mu_1^\circ\rangle \doteq [|\mu_1^\circ\rangle]_\gamma = \begin{bmatrix} 9.9992\text{E}{-1} \\ -1.2564\text{E}{-2} \\ 6.9815\text{E}{-4} \\ 8.7721\text{E}{-6} \end{bmatrix};$$

$$|\mu_2^\circ\rangle \doteq [|\mu_2^\circ\rangle]_\gamma = \begin{bmatrix} 1.2564\text{E}{-2} \\ 9.9992\text{E}{-1} \\ 8.7721\text{E}{-6} \\ -6.9815\text{E}{-4} \end{bmatrix};$$

$$|\mu_3^\circ\rangle \doteq [|\mu_3^\circ\rangle]_\gamma = \begin{bmatrix} -6.9815\text{E}{-4} \\ 8.7721\text{E}{-6} \\ 9.9992\text{E}{-1} \\ 1.2564\text{E}{-2} \end{bmatrix};$$

$$|\mu_4^\circ\rangle \doteq [|\mu_4^\circ\rangle]_\gamma = \begin{bmatrix} 8.7721\text{E}{-6} \\ 6.9815\text{E}{-4} \\ -1.2564\text{E}{-2} \\ 9.9992\text{E}{-1} \end{bmatrix}.$$

Channel matrix $[P(j|i)] = [\langle\psi_i|\hat{\Pi}_j^\circ|\psi_i\rangle]$:

$$\begin{bmatrix} 9.9607\text{E}{-1} & 3.9274\text{E}{-3} & 3.6390\text{E}{-6} & 1.4348\text{E}{-8} \\ 1.4156\text{E}{-3} & 9.9607\text{E}{-1} & 2.5142\text{E}{-3} & 1.3116\text{E}{-6} \\ 1.3116\text{E}{-6} & 2.5142\text{E}{-3} & 9.9607\text{E}{-1} & 1.4156\text{E}{-3} \\ 1.4348\text{E}{-8} & 3.6390\text{E}{-6} & 3.9274\text{E}{-3} & 9.9607\text{E}{-1} \end{bmatrix}.$$

(Note that the diagonal entries are identical.)
Minimal average probability of error, $\bar{P}_{\rm e}^\circ$:

$$\bar{P}_{\rm e}^\circ = 3.9311\text{E}{-3}.$$

<u>**case of** $\kappa = 0.3$</u>
Optimal distribution $\mathbf{p}^\circ$:

$$p_1^\circ = 0.18793;$$
$$p_2^\circ = 0.31207;$$
$$p_3^\circ = 0.31207;$$
$$p_4^\circ = 0.18793.$$

Detection vectors:

$$|\mu_1^\circ\rangle \doteq [|\mu_1^\circ\rangle]_\gamma = \begin{bmatrix} 9.9921\text{E}{-1} \\ -3.9396\text{E}{-2} \\ 5.9399\text{E}{-3} \\ 2.3419\text{E}{-4} \end{bmatrix};$$

$$|\mu_2^\circ\rangle \doteq [|\mu_2^\circ\rangle]_\gamma = \begin{bmatrix} 3.9396\text{E}{-2} \\ 9.9921\text{E}{-1} \\ 2.3419\text{E}{-4} \\ -5.9399\text{E}{-3} \end{bmatrix};$$

$$|\mu_3^\circ\rangle \doteq [|\mu_3^\circ\rangle]_\gamma = \begin{bmatrix} -5.9399\text{E}{-3} \\ 2.3419\text{E}{-4} \\ 9.9921\text{E}{-1} \\ 3.9396\text{E}{-2} \end{bmatrix};$$

$$|\mu_4^\circ\rangle \doteq [|\mu_4^\circ\rangle]_\gamma = \begin{bmatrix} 2.3419\text{E}{-4} \\ 5.9399\text{E}{-3} \\ -3.9396\text{E}{-2} \\ 9.9921\text{E}{-1} \end{bmatrix}.$$

Channel matrix $[P(j|i)] = [\langle\psi_i|\hat{\Pi}_j^\circ|\psi_i\rangle]$:

$$\begin{bmatrix} 9.6287\text{E}{-1} & 3.6936\text{E}{-2} & 1.9196\text{E}{-4} & 7.4183\text{E}{-6} \\ 1.3396\text{E}{-2} & 9.6287\text{E}{-1} & 2.3670\text{E}{-2} & 6.96210\text{E}{-5} \\ 6.9621\text{E}{-5} & 2.3670\text{E}{-2} & 9.6287\text{E}{-1} & 1.3396\text{E}{-2} \\ 7.4183\text{E}{-6} & 1.9196\text{E}{-4} & 3.6936\text{E}{-2} & 9.6287\text{E}{-1} \end{bmatrix}.$$

Minimal average probability of error, $\bar{P}_{\rm e}^\circ$:

$$\bar{P}_{\rm e}^\circ = 3.7135\text{E}{-2}.$$

<u>**case of** $\kappa = 0.5$</u>
Optimal distribution $\mathbf{p}^\circ$:

$$p_1^\circ = 0.18789;$$
$$p_2^\circ = 0.31211;$$

$$p_3^\circ = 0.31211;$$

$$p_4^\circ = 0.18789.$$

Detection vectors:

$$|\mu_1^\circ\rangle \doteq [|\mu_1^\circ\rangle]_\gamma = \begin{bmatrix} 9.9725\text{E}{-1} \\ -7.2632\text{E}{-2} \\ 1.4557\text{E}{-2} \\ 1.0602\text{E}{-3} \end{bmatrix};$$

$$|\mu_2^\circ\rangle \doteq [|\mu_2^\circ\rangle]_\gamma = \begin{bmatrix} 7.2632\text{E}{-2} \\ 9.9725\text{E}{-1} \\ 1.0602\text{E}{-3} \\ -1.4557\text{E}{-2} \end{bmatrix};$$

$$|\mu_3^\circ\rangle \doteq [|\mu_3^\circ\rangle]_\gamma = \begin{bmatrix} -1.4557\text{E}{-2} \\ 1.0602\text{E}{-3} \\ 9.9725\text{E}{-1} \\ 7.2632\text{E}{-2} \end{bmatrix};$$

$$|\mu_4^\circ\rangle \doteq [|\mu_4^\circ\rangle]_\gamma = \begin{bmatrix} 1.0602\text{E}{-3} \\ 1.4557\text{E}{-2} \\ -7.2632\text{E}{-2} \\ 9.9725\text{E}{-1} \end{bmatrix}.$$

Channel matrix $[P(j|i)] = [\langle\psi_i|\hat{\Pi}_j^\circ|\psi_i\rangle]$:

$$\begin{bmatrix} 8.8754\text{E}{-1} & 1.1204\text{E}{-1} & 3.6203\text{E}{-4} & 6.0792\text{E}{-5} \\ 4.0607\text{E}{-2} & 8.8754\text{E}{-1} & 7.1726\text{E}{-2} & 1.3121\text{E}{-4} \\ 1.3121\text{E}{-4} & 7.1726\text{E}{-2} & 8.8754\text{E}{-1} & 4.0607\text{E}{-2} \\ 6.0792\text{E}{-5} & 3.6203\text{E}{-4} & 1.1204\text{E}{-1} & 8.8754\text{E}{-1} \end{bmatrix}.$$

Minimal average probability of error, $\bar{P}_\text{e}^\circ$:

$$\bar{P}_\text{e}^\circ = 0.11246.$$

<u>case of $\kappa = 0.7$</u>

Optimal distribution $\mathbf{p}^\circ$:

$$p_1^\circ = 0.18750;$$

$$p_2^\circ = 0.31250;$$

$$p_3^\circ = 0.31250;$$

$$p_4^\circ = 0.18750.$$

Detection vectors:

$$|\mu_1^\circ\rangle \doteq [|\mu_1^\circ\rangle]_\gamma = \begin{bmatrix} 9.9281\text{E}{-1} \\ -1.1798\text{E}{-1} \\ 2.0250\text{E}{-2} \\ 2.4063\text{E}{-3} \end{bmatrix};$$

$$|\mu_2^\circ\rangle \doteq [|\mu_2^\circ\rangle]_\gamma = \begin{bmatrix} 1.1798\text{E}{-1} \\ 9.9281\text{E}{-1} \\ 2.4063\text{E}{-3} \\ -2.0250\text{E}{-2} \end{bmatrix};$$

$$|\mu_3^\circ\rangle \doteq [|\mu_3^\circ\rangle]_\gamma = \begin{bmatrix} -2.0250\text{E}{-2} \\ 2.4063\text{E}{-3} \\ 9.9281\text{E}{-1} \\ 1.1798\text{E}{-1} \end{bmatrix};$$

$$|\mu_4^\circ\rangle \doteq [|\mu_4^\circ\rangle]_\gamma = \begin{bmatrix} 2.4063\text{E}{-3} \\ 2.0250\text{E}{-2} \\ -1.1798\text{E}{-1} \\ 9.9281\text{E}{-1} \end{bmatrix}.$$

Channel matrix $[P(j|i)] = [\langle\psi_i|\hat{\Pi}_j^\circ|\psi_i\rangle]$:

$$\begin{bmatrix} 7.5379\text{E}{-1} & 2.4527\text{E}{-1} & 9.0039\text{E}{-4} & 3.7492\text{E}{-5} \\ 8.8304\text{E}{-2} & 7.5379\text{E}{-1} & 1.5758\text{E}{-1} & 3.2416\text{E}{-4} \\ 3.2416\text{E}{-4} & 1.5758\text{E}{-1} & 7.5379\text{E}{-1} & 8.8304\text{E}{-2} \\ 3.7492\text{E}{-5} & 9.0034\text{E}{-4} & 2.4527\text{E}{-1} & 7.5379\text{E}{-1} \end{bmatrix}.$$

Minimal average probability of error, $\bar{P}_\text{e}^\circ$:

$$\bar{P}_\text{e}^\circ = 0.24621.$$

<u>case of $\kappa = 0.9$</u>

Optimal distribution $\mathbf{p}^\circ$:

$$p_1^\circ = 0.19516;$$

$$p_2^\circ = 0.30484;$$

$$p_3^\circ = 0.30484;$$

$$p_4^\circ = 0.19516.$$

Detection vectors:

$$|\mu_1^\circ\rangle \doteq [|\mu_1^\circ\rangle]_\gamma = \begin{bmatrix} 9.8789\text{E}{-1} \\ -1.5517\text{E}{-1} \\ -6.8750\text{E}{-4} \\ -1.0799\text{E}{-4} \end{bmatrix};$$

$$|\mu_2^\circ\rangle \doteq [|\mu_2^\circ\rangle]_\gamma = \begin{bmatrix} 1.5517\text{E}{-1} \\ 9.8789\text{E}{-1} \\ -1.0799\text{E}{-4} \\ 6.8750\text{E}{-1} \end{bmatrix};$$

$$|\mu_3^\circ\rangle \doteq [|\mu_3^\circ\rangle]_\gamma = \begin{bmatrix} 6.8750\text{E}{-4} \\ -1.0799\text{E}{-4} \\ 9.8789\text{E}{-1} \\ 1.5517\text{E}{-1} \end{bmatrix};$$

$$|\mu_4^\circ\rangle \doteq [|\mu_4^\circ\rangle]_\gamma = \begin{bmatrix} -1.0799\text{E}{-4} \\ -6.8750\text{E}{-4} \\ -1.5517\text{E}{-1} \\ 9.8789\text{E}{-1} \end{bmatrix}.$$

Channel matrix $[P(j|i)] = [\langle\psi_i|\hat{\Pi}_j^\circ|\psi_i\rangle]$:

$$\begin{bmatrix} 5.2912\text{E}{-1} & 4.0485\text{E}{-1} & 6.4013\text{E}{-2} & 2.0242\text{E}{-3} \\ 1.6594\text{E}{-1} & 5.2912\text{E}{-1} & 2.7871\text{E}{-1} & 2.6238\text{E}{-2} \\ 2.6238\text{E}{-2} & 2.7871\text{E}{-1} & 5.2912\text{E}{-1} & 1.6594\text{E}{-1} \\ 2.0242\text{E}{-3} & 6.4013\text{E}{-2} & 4.0485\text{E}{-1} & 5.2912\text{E}{-1} \end{bmatrix}.$$

Minimal average probability of error, $\bar{P}_\text{e}^\circ$:

$$\bar{P}_\text{e}^\circ = 0.47089.$$

G. Optimal detection of QPSK

The quadrature phase-shift keying (QPSK) coherent state signal is defined as

$$\mathcal{S} = \{|\alpha\rangle, |\mathrm{i}\alpha\rangle, |-\alpha\rangle, |-\mathrm{i}\alpha\rangle\}.$$

The average number of signal photons for QPSK is given as $\bar{n}_\mathrm{s} = |\alpha|^2$. The optimality of the SRM for the symmetric pure state signal (including PSK coherent state signal) was proved by Belavkin [6] and by Ban *et al.* [8], independently.

The Gram matrix of QPSK is given as follows [21].

$$\mathsf{G} = \begin{bmatrix} 1 & Z_\mathrm{c}+\mathrm{i}Z_\mathrm{s} & \zeta^2 & Z_\mathrm{c}-\mathrm{i}Z_\mathrm{s} \\ Z_\mathrm{c}-\mathrm{i}Z_\mathrm{s} & 1 & Z_\mathrm{c}+\mathrm{i}Z_\mathrm{s} & \zeta^2 \\ \zeta^2 & Z_\mathrm{c}-\mathrm{i}Z_\mathrm{s} & 1 & Z_\mathrm{c}+\mathrm{i}Z_\mathrm{s} \\ Z_\mathrm{c}+\mathrm{i}Z_\mathrm{s} & \zeta^2 & Z_\mathrm{c}-\mathrm{i}Z_\mathrm{s} & 1 \end{bmatrix},$$

where $\zeta = \exp[-|\alpha|^2]$, $Z_\mathrm{c} = \zeta\cos[|\alpha|^2]$, and $Z_\mathrm{s} = \zeta\sin[|\alpha|^2]$. Since this matrix is a circular matrix, its eigenvalues and eigenvectors are respectively given as follows.

$$\lambda_1^{\mathrm{QPSK}} = 1+\zeta^2-2Z_\mathrm{c}, \quad \vec{\lambda}_1^{\mathrm{QPSK}} = \frac{1}{2}\begin{bmatrix} -1 \\ 1 \\ -1 \\ 1 \end{bmatrix};$$

$$\lambda_2^{\mathrm{QPSK}} = 1+\zeta^2+2Z_\mathrm{c}, \quad \vec{\lambda}_2^{\mathrm{QPSK}} = \frac{1}{2}\begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix};$$

$$\lambda_3^{\mathrm{QPSK}} = 1-\zeta^2-2Z_\mathrm{s}, \quad \vec{\lambda}_3^{\mathrm{QPSK}} = \frac{1}{2}\begin{bmatrix} \mathrm{i} \\ -1 \\ -\mathrm{i} \\ 1 \end{bmatrix};$$

$$\lambda_4^{\mathrm{QPSK}} = 1-\zeta^2+2Z_\mathrm{s}, \quad \vec{\lambda}_4^{\mathrm{QPSK}} = \frac{1}{2}\begin{bmatrix} -\mathrm{i} \\ -1 \\ \mathrm{i} \\ 1 \end{bmatrix}.$$

From these, the minimal average probability of error at the uniform distribution of signal elements is given as follows (See also [9]).

$$\bar{P}_\mathrm{e}(\mathrm{QPSK}) = 1 - \frac{1}{16}\left(\sum_{i=1}^{4}\sqrt{\lambda_i^{\mathrm{QPSK}}}\right)^2. \tag{67}$$

Here let us compare the error rate performance of QASK coherent state signal with that of QPSK. Since QPSK coherent state signal is a symmetric pure state signal, its SRM is identical to the Bayes-optimal detection at the uniform distribution of signal elements and to the minimax detection. The simulation results are shown in Fig. 7. When the designed error probability is set to $\bar{P}_\mathrm{e} = 10^{-5}$, QASK requires 13.2 photons, while QPSK requires 5.4 photons. Thus, 3.9dB power budget of QPSK is expected than QASK at $\bar{P}_\mathrm{e} = 10^{-5}$. In an ordinary usage of the optical communications system, QPSK coherent state signal is clearly better than QASK. However, QASK might have a potential than QPSK in some cases of unusual usage of an optical communications system. Such applications of QASK coherent state signal will be discussed else where.

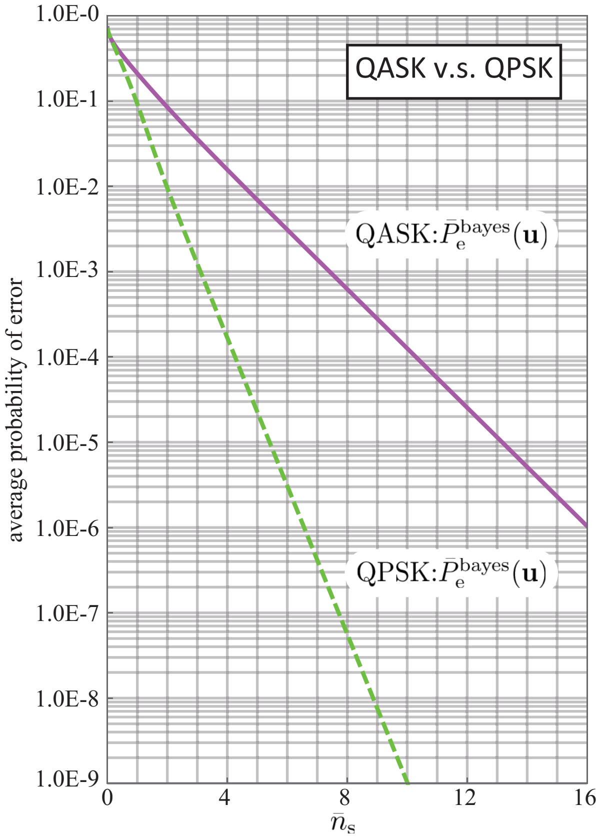

Fig. 7. QASK v.s. QPSK