ISSN 2186-6570

A characterization of von Neumann entropy using functors

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Tamagawa University Quantum ICT Research Institute Bulletin, Vol.13, No.1, 1-5, 2023

Tamagawa University Quantum ICT Research Institute Bulletin
Vol.13 No.1 : 1–5 (2023)

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***Abstract*—Baez, Fritz, and Leinster derived a method for characterizing Shannon entropy in classical systems. In this method, they considered a functor from a certain category to the monoid of non-negative real numbers with addition as a map from measure-preserving functions to non-negative real numbers, and derived Shannon entropy by imposing several simple conditions. We propose a method for characterizing von Neumann entropy by extending their results to quantum systems.**

I. Introduction

Von Neumann entropy is a key concept in quantum information theory, which quantifies the ambiguity in quantum states. Also, Shannon entropy, an important concept in classical information theory, can be regarded as von Neumann entropy in classical states. Baez, Fritz, and Leinster derived Shannon entropy as a quantity that characterizes measure-preserving functions from classical systems to classical systems [1]. Specifically, they showed that if a map from such measure-preserving functions with probability measures to non-negative real numbers is regarded as a functor in category theory and satisfies certain properties, it is expressed as the difference of Shannon entropy. In this paper, we try to derive von Neumann entropy (or Segal entropy) by extending their result to quantum systems. Parzygnat has recently extended their result [2]. The main difference of our method compared to the one of Ref. [2] is the use of conditions that are considered weaker. In Refs. [1] and [2], the discussion was limited to measure-preserving functions (or their extensions to quantum systems, unital *-homomorphisms), but in this paper, we consider quantities characterizing any quantum channel. Although not mentioned in this paper, many different approaches are known for characterizing Shannon entropy and von Neumann entropy (e.g., [3]–[6]).

II. Previous research

A. Preliminaries

Let $\mathbb{N}$, $\mathbb{R}$, and $\mathbb{C}$ denote the sets of natural numbers, real numbers, and complex numbers, respectively. Also, let $[0,1]$ be the set of real numbers between 0 and 1 inclusive.

In this paper, we refer to a finite-dimensional quantum system with decoherence as a quantum system, or simply a system. Any system A can be represented in the following form:

$$A = \bigoplus_{i=1}^{k} A_i, \quad A_i \cong \mathsf{M}_{n_i}, \tag{1}$$

where M_n is the set of complex square matrices of order n and n_i is a natural number determined by the subsystem A_i. The set of states (i.e., density operators) of a system A, which is always convex, is denoted by St_A, and the set of channels (i.e., trace-preserving completely positive maps) from a system A to a system B is denoted by $\mathsf{Chn}(A,B)$. A state is called pure if it is an extreme point of the convex set St_A.

Example 1: In the case of $k = 1$, we have $A \cong \mathsf{M}_{n_1}$. St_A is isomorphic to the set of density matrices of order n_1.

Example 2: When $n_1 = \cdots = n_k = 1$, we call a system A a classical system and its states classical states. We often represent a classical system A as $\mathbb{C}^X$, where X is a finite set with $|X| = k$ elements. A classical system $\mathbb{C}^X$ satisfies $\mathbb{C}^X \cong \bigoplus_{i=1}^{|X|} \mathsf{M}_1 \cong \mathbb{C}^{|X|}$. $\mathsf{St}_{\mathbb{C}^X}$ is isomorphic to the set of diagonal density matrices of order $|X|$, which is also isomorphic to the set of $|X|$-dimensional non-negative row vectors whose sum of components are 1. For example, we have

$$\begin{aligned}\mathsf{St}_{\mathbb{C}^2} &\cong \left\{ \begin{bmatrix} p_1 & 0 \\ 0 & p_2 \end{bmatrix} \middle| \ p_1, p_2 \ge 0, \ p_1 + p_2 = 1 \right\} \\ &\cong \left\{ \begin{bmatrix} p_1 \\ p_2 \end{bmatrix} \middle| \ p_1, p_2 \ge 0, \ p_1 + p_2 = 1 \right\}.\end{aligned}$$

Each state can also be represented as a collection $\{p_x\}_{x\in X}$ of non-negative real numbers satisfying $\sum_{x\in X} p_x = 1$, which can be regarded as a probability distribution. In particular, when $|X| = k = 1$, we often simply write $\mathbb{C}$, by abuse of notation. Without loss of generality, we may assume $\mathsf{St}_{\mathbb{C}} = \{1\}$. There are $|X|$ pure states in the classical system $\mathbb{C}^X$, which we will denote by ϕ_x^X $(x \in X)$. For example, when $|X| = 2$, there are two pure states represented by $[1,0]^\mathsf{T}$ and $[0,1]^\mathsf{T}$, where $^\mathsf{T}$ denotes transposition. The state p of the classical system $\mathbb{C}^X$ can be expressed in the form

$$p = \sum_{x\in X} p_x \phi_x^X, \quad p_x \ge 0 \ (\forall x \in X), \ \sum_{x\in X} p_x = 1.$$

A channel f from a system A to a system B is called pure-to-pure[1] if it maps any pure state to a pure state. For each system A, the map

$$\mathsf{St}_A \ni \omega \mapsto \mathrm{Tr}\,\omega = 1 \in \mathsf{St}_{\mathbb{C}}$$

is the unique channel from A to $\mathbb{C}$. This channel, denoted by Tr^A, is pure-to-pure.

Let $\mathsf{FinProb}$ be the following category:

- Its each object is a pair $(\mathbb{C}^X, p)$ of a classical system $\mathbb{C}^X$ and its state $p \in \mathsf{St}_{\mathbb{C}^X}$.
- Its each morphism from an object $(\mathbb{C}^X, p)$ to an object $(\mathbb{C}^Y, q)$ is a pure-to-pure channel f from $\mathbb{C}^X$ to $\mathbb{C}^Y$ such that $q = f \circ p$. To indicate that the domain of this morphism is $(\mathbb{C}^X, p)$, we write f_p instead of f.
- The composite of its morphisms is the composite of channels as maps, and its each identity morphism is the identity channel.

Also, let $\mathbb{BR}$ be the following category:

- It has a single object.
- Its each morphism is a real number.
- The composite of morphisms is the sum of real numbers, and its identity morphism is 0.

Let $\mathbb{BR}_+$ be the subcategory of $\mathbb{BR}$ restricted to non-negative numbers as morphisms.

For any two classical systems $\mathbb{C}^X$ and $\mathbb{C}^Y$, the classical system $\mathbb{C}^{X \sqcup Y}$ (where $\sqcup$ denotes disjoint union) can be considered as their direct sum. The state r of $\mathbb{C}^{X \sqcup Y}$ can be expressed in the form

$$r = \lambda p \oplus (1-\lambda) q := \sum_{x \in X} \lambda p_x \phi_x^{X \sqcup Y} + \sum_{y \in Y} (1-\lambda) q_y \phi_y^{X \sqcup Y}$$

using some $p \in \mathsf{St}_{\mathbb{C}^X}$, $q \in \mathsf{St}_{\mathbb{C}^Y}$, and $\lambda \in [0,1]$. The set of pure states of $\mathbb{C}^{X \sqcup Y}$ is $\{\phi_x^{X \sqcup Y}\}_{x \in X} \sqcup \{\phi_y^{X \sqcup Y}\}_{y \in Y}$. For any two channels $f \in \mathsf{Chn}(\mathbb{C}^X, \mathbb{C}^{X'})$ and $g \in \mathsf{Chn}(\mathbb{C}^Y, \mathbb{C}^{Y'})$[2], the channel defined by the map

$$\begin{aligned} &\mathsf{St}_{\mathbb{C}^{X \sqcup Y}} \ni \lambda p \oplus (1-\lambda) q \\ &\quad \mapsto \lambda (f \circ p) \oplus (1-\lambda)(g \circ q) \in \mathsf{St}_{\mathbb{C}^{X' \sqcup Y'}} \end{aligned}$$

is denoted by $f \oplus g \in \mathsf{Chn}(\mathbb{C}^{X \sqcup Y}, \mathbb{C}^{X' \sqcup Y'})$. If f and g are pure-to-pure, then so is $f \oplus g$.

A functor H from $\mathsf{FinProb}$ to $\mathbb{BR}$ is said to be continuous if, for any two classical systems $\mathbb{C}^X$ and $\mathbb{C}^Y$ and any sequence $\mathbb{N} \ni n \mapsto f^{(n)}_{p^{(n)}} : (\mathbb{C}^X, p^{(n)}) \to (\mathbb{C}^Y, f^{(n)} \circ p^{(n)})$ converging to a morphism $f_p : (\mathbb{C}^X, p) \to (\mathbb{C}^Y, f \circ p)$, the sequence $\mathbb{N} \ni n \mapsto H(f^{(n)}_{p^{(n)}})$ converges to the morphism $H(f_p)$. Note that we can consider the convergence of a sequence of pure-to-pure channels; indeed, any pure-to-pure channel from $\mathbb{C}^X$ to $\mathbb{C}^Y$ belongs to the normed vector space $\mathbb{C}^{X \times Y}$, which has a metric $d(f,g) := \|f - g\|$ (where $f, g \in \mathbb{C}^{X \times Y}$).

[1] In Ref. [1], a pure-to-pure channel is called a measurement-preserving function.

[2] When we say "for any $f \in \mathsf{Chn}(\mathbb{C}^X, \mathbb{C}^{X'})$", unless otherwise stated, we assume that $\mathbb{C}^X$ and $\mathbb{C}^{X'}$ are also arbitrary.

B. Baez, Fritz, and Leinster's theorem

Baez, Fritz, and Leinster proved the following theorem [1] (expressed in the above notation):

Theorem 1 (Baez, Fritz, and Leinster (BFL) [1]): Assume that a functor H_{BFL} from $\mathsf{FinProb}$ to $\mathbb{BR}_+$ satisfies the following conditions:

1) **Continuity:** H_{BFL} is continuous.
2) **Convex linearity:** For any two pure-to-pure channels $f \in \mathsf{Chn}(\mathbb{C}^X, \mathbb{C}^{X'})$ and $g \in \mathsf{Chn}(\mathbb{C}^Y, \mathbb{C}^{Y'})$ and any $p \in \mathsf{St}_{\mathbb{C}^X}$, $q \in \mathsf{St}_{\mathbb{C}^Y}$, and $\lambda \in [0,1]$, $H_{\mathrm{BFL}}((f \oplus g)_{\lambda p \oplus (1-\lambda) q}) = \lambda H_{\mathrm{BFL}}(f_p) + (1-\lambda) H_{\mathrm{BFL}}(g_q)$ holds.

Then, there exists a non-negative real number c such that

$$H_{\mathrm{BFL}}(f_p) = c(\mathrm{S}_{\mathrm{Sh}}(p) - \mathrm{S}_{\mathrm{Sh}}(f \circ p)), \tag{2}$$

where f_p is any morphism from $(\mathbb{C}^X, p)$ to $(\mathbb{C}^Y, f \circ p)$ in $\mathsf{FinProb}$, and $\mathrm{S}_{\mathrm{Sh}}(p)$ is the Shannon entropy of p, i.e., $\mathrm{S}_{\mathrm{Sh}}(p) := -\sum_{x \in X} p_x \log p_x$, where we set for convenience $0 \log 0 = 0$.

C. Parzygnat's theorem

Parzygnat showed that Segal entropy (or von Neumann entropy) can be derived by extending BFL's theorem (Theorem 1) to quantum systems [2]. We introduce Parzygnat's results after some preparation. Note that since it is difficult to concisely state Parzygnat's theorem in a self-contained manner, this paper omits explanations of some terms (for details, see [2] and the references cited therein).

When a system A is expressed in the form of Eq. (1), its state ω can be expressed in the form $\omega = \sum_{i=1}^k p_i \omega_i$ ($\omega_i \in \mathsf{St}_{A_i}$). When expressed in this way, the Segal entropy, $\mathrm{S}_{\mathrm{Se}}(\omega)$, of ω is defined by

$$\begin{aligned} \mathrm{S}_{\mathrm{Se}}(\omega) &:= \mathrm{S}_{\mathrm{Se}}(p) + \sum_{i=1}^k p_i \mathrm{S}_{\mathrm{vN}}(\omega_i) \\ &= -\sum_{i=1}^k \mathrm{Tr}(p_i \omega_i \log(p_i \omega_i)), \end{aligned}$$

where S_{vN} is von Neumann entropy, i.e., $\mathrm{S}_{\mathrm{vN}}(\rho) := -\mathrm{Tr}(\rho \log \rho)$. The Segal entropy $\mathrm{S}_{\mathrm{Se}}(\omega)$ is equal to the von Neumann entropy $\mathrm{S}_{\mathrm{vN}}(\tilde{\omega})$ when the state $\omega \in \mathsf{St}_A$ is represented as a density matrix $\tilde{\omega}$ of order $n = \sum_{i=1}^k n_i$. Therefore, it can be said that there is no substantial difference between Segal entropy and von Neumann entropy for finite-dimensional systems (while a noticeable difference arises for infinite-dimensional systems). Also, for any state $p \in \mathsf{St}_{\mathbb{C}^X}$ of a classical system, we have $\mathrm{S}_{\mathrm{Se}}(p) = \mathrm{S}_{\mathrm{Sh}}(p)$.

A finite-dimensional C^*-algebra A can be expressed in the form of Eq. (1) [7]. We denote a state of such a C^*-algebra A by adding a symbol * such as ω^*. Each state ω^* of A can be expressed as a map

$$\omega^* : A \ni Q \mapsto \mathrm{Tr}(\omega Q) \in \mathbb{C}$$

using a density operator $\omega \in \mathsf{St}_A$. For each state ω^*, we write ω for the density operator satisfying this equation. Note that when referring to a state of a C^*-algebra A, it means a map from A to $\mathbb{C}$, not an element of St_A.

Let NCFinProb be the following category:

- Its each object is a pair (A, ω^*) of a C^*-algebra A and its state ω^*.
- Its each morphism from an object (A, ω^*) to an object (B, ξ^*) is a state-preserving unital *-homomorphism f from B to A such that $\xi^* = \omega^* \circ f$. To indicate that the domain of this morphism is (A, ω^*), we write f_{ω^*} instead of f.
- The composite of its morphisms is the composite of maps, and its each identity morphism is the identity map.

Parzygnat proved the following theorem:

Theorem 2 (Parzygnat [2]): Assume that a functor H from NCFinProb to $\mathbb{BR}$ satisfies the following conditions:

1) H is continuous.
2) $H(!^A_{\omega^*}) \geq 0$ holds for any state ω^* of a C^*-algebra A, with equality for a pure state, where $!^A$ is the unique unital *-homomorphism from $\mathbb{C}$ to A.
3) H is a fibred functor from the fibration NCFinProb $\to$ fdC*-Alg to the fibration $\mathbb{BR} \to \underline{\mathbf{1}}$.
4) H is orthogonally affine.

Then, there exists a non-negative real number c such that

$$H(f_{\omega^*}) = c(\mathrm{S}_{\mathrm{Se}}(\omega) - \mathrm{S}_{\mathrm{Se}}(\xi))$$

holds, where f_{ω^*} is any morphism from (A, ω^*) to $(B, \xi^* := \omega^* \circ f)$ in NCFinProb.

In this theorem, Condition 3) defines a functor H as a (Grothendieck) fibration. This condition seems to make the relationship that should hold between the fibrations NCFinProb $\to$ fdC*-Alg and $\mathbb{BR} \to \underline{\mathbf{1}}$ more explicit. Condition 1) of this theorem and Condition 1) of BFL's theorem (i.e., Theorem 1) are essentially the same[3]. Furthermore, Condition 2) is closely related to the condition of BFL's theorem that H_{BFL} is a functor from FinProb to $\mathbb{BR}_+$, and Condition 4) is closely related to Condition 2) of BFL's theorem.

In a categorical sense, the four conditions of Theorem 2 seem to be elegant, but they may seem somewhat too strong intuitively. In what follows, we try to derive Segal entropy (or von Neumann entropy) from different conditions, which seem intuitively weaker than the conditions of Theorem 2.

III. Main theorem

Let FinState be the following category:

- Its each object is a pair (A, ω) of a system A and its state $\omega \in \mathsf{St}_A$.
- Its each morphism from an object (A, ω) to an object (B, ξ) is a channel f from A to B such that $\xi = f \circ \omega$. To indicate that the domain of this morphism is (A, ω), we write f_ω instead of f.
- The composite of its morphisms is the composite of channels as maps, and its each identity morphism is the identity channel.

Note that the morphisms of FinState are not limited to pure-to-pure channels (or unital *-homomorphisms). FinProb is a subcategory of FinState. For any two channels $f \in \mathsf{Chn}(A, B)$ and $g \in \mathsf{Chn}(B, C)$ and any $\omega \in \mathsf{St}_A$, the composite of morphisms $f_\omega : (A, \omega) \to (B, f \circ \omega)$ and $g_{f\circ\omega} : (B, f \circ \omega) \to (C, g \circ f \circ \omega)$ is

$$g_{f\circ\omega} \circ f_\omega = (g \circ f)_\omega : (A, \omega) \to (C, g \circ f \circ \omega). \quad (3)$$

A channel f from a system A to a system B is called left-invertible (or split mono) if there exists a channel g from B to A such that $g \circ f$ is the identity channel.

As in the case of classical systems, for any two systems A and B, their direct sum $A \oplus B$ can be considered. Each state σ of $A \oplus B$ can be expressed in the form

$$\sigma = \lambda\omega \oplus (1 - \lambda)\xi$$

using some $\omega \in \mathsf{St}_A$, $\xi \in \mathsf{St}_B$, and $\lambda \in [0, 1]$. The states $\omega \oplus 0$ and $0 \oplus \xi$ with zero operators 0 are orthogonal. For any two channels $f \in \mathsf{Chn}(A, A')$ and $g \in \mathsf{Chn}(B, B')$, we write the channel defined by the map

$$\begin{aligned} \mathsf{St}_{A\oplus B} &\ni \lambda\omega \oplus (1 - \lambda)\xi \\ &\mapsto \lambda(f \circ \omega) \oplus (1 - \lambda)(g \circ \xi) \in \mathsf{St}_{A'\oplus B'} \end{aligned}$$

as $f \oplus g \in \mathsf{Chn}(A \oplus B, A' \oplus B')$. If f and g are pure-to-pure, then so is $f \oplus g$.

A functor H from FinState to $\mathbb{BR}$ is said to be continuous if, for any two systems A and B and any sequence $\mathbb{N} \ni n \mapsto f^{(n)}_{\omega^{(n)}} : (A, \omega^{(n)}) \to (B, f^{(n)} \circ \omega^{(n)})$ converging to $f_\omega : (A, \omega) \to (B, f \circ \omega)$, the sequence $\mathbb{N} \ni n \mapsto H(f^{(n)}_{\omega^{(n)}})$ converges to $H(f_\omega)$.

For a given functor H from FinState to $\mathbb{BR}$, let

$$S(\omega) := H(\mathrm{Tr}^A_\omega), \quad (4)$$

where $\mathrm{Tr}^A_\omega : (A, \omega) \to (\mathbb{C}, \mathrm{Tr}\,\omega = 1)$ is the morphism corresponding to the unique channel Tr^A from A to $\mathbb{C}$.

In this paper, we claim that the following theorem holds as an extension of BFL's theorem to quantum systems:

Theorem 3 (main): Assume that a functor H from FinState to $\mathbb{BR}$ satisfies the following conditions:

1) **Continuity:** H is continuous.
2) **Convex linearity:** For any two pure-to-pure channels $f \in \mathsf{Chn}(A, A')$ and $g \in \mathsf{Chn}(B, B')$ and any $\omega \in \mathsf{St}_A$, $\xi \in \mathsf{St}_B$, and $\lambda \in [0, 1]$, $H((f \oplus g)_{\lambda\omega\oplus(1-\lambda)\xi}) = \lambda H(f_\omega) + (1 - \lambda)H(g_\xi)$ holds.
3) **Positivity for pure-to-pure channels:** $H(f_\omega) \geq 0$

[3] By considering the inclusion functor I: $\mathsf{FinProb}^{\mathrm{op}} \to$ NCFinProb from the opposite category $\mathsf{FinProb}^{\mathrm{op}}$ of FinProb as a subcategory of NCFinProb, we can regard the functor H_{BFL} in BFL's theorem as the composite functor HI: $\mathsf{FinProb}^{\mathrm{op}} \to \mathbb{BR}$.

holds for any pure-to-pure channel $f \in \mathsf{Chn}(A, B)$ and any $\omega \in \mathsf{St}_A$ (in which case, f_ω is a morphism from (A, ω) to $(B, f \circ \omega)$), with equality for a left-invertible channel f.

Then, there exists a non-negative real number c such that

$$H(f_\omega) = c(\mathrm{S}_{\mathrm{Se}}(\omega) - \mathrm{S}_{\mathrm{Se}}(f \circ \omega)), \tag{5}$$

where f_ω is any morphism from (A, ω) to $(B, f \circ \omega)$ in $\mathsf{FinState}$.

Note that Conditions 2) and 3) can be weakened as follows:

2') **Convex linearity:** For any two pure-to-pure channels $f \in \mathsf{Chn}(\mathbb{C}^X, \mathbb{C}^{X'})$ and $g \in \mathsf{Chn}(\mathbb{C}^Y, \mathbb{C}^{Y'})$ from classical systems to classical systems and any $p \in \mathsf{St}_{\mathbb{C}^X}$, $q \in \mathsf{St}_{\mathbb{C}^Y}$, and $\lambda \in [0, 1]$, $H((f \oplus g)_{\lambda p \oplus (1-\lambda) q}) = \lambda H(f_p) + (1 - \lambda) H(g_q)$ holds.

3') **Positivity for pure-to-pure channels:** $H(f_p) \geq 0$ holds for any pure-to-pure channel $f \in \mathsf{Chn}(\mathbb{C}^X, B)$ from a classical system $\mathbb{C}^X$ and any $p \in \mathsf{St}_{\mathbb{C}^X}$ (in which case, f_p is a morphism from $(\mathbb{C}^X, p)$ to $(B, f \circ p)$), with equality for a left-invertible channel f.

We discuss the relationship between this theorem and BFL's theorem (Theorem 1). Let I be the inclusion functor from subcategory $\mathsf{FinProb}$ of $\mathsf{FinState}$ to $\mathsf{FinState}$; then, we can say that the functor H_{BFL} in BFL's theorem is the composite $HI \colon \mathsf{FinProb} \to \mathbb{BR}$. In this case, from Conditions 1) and 2') of the main theorem, Conditions 1) and 2) of BFL's theorem can be obtained. Conditions 1) and 2') rephrase the corresponding conditions of BFL's theorem in the terms of the functor H instead of $H_{\mathrm{BFL}} = HI$. Also, Condition 3') of the main theorem corresponds to the condition in BFL's theorem that H_{BFL} is a functor from $\mathsf{FinProb}$ to $\mathbb{BR}_+$, and the former condition is stronger than the latter. In fact, from Condition 3') of the main theorem, it is clear that $H_{\mathrm{BFL}} = HI$ maps any morphism of $\mathsf{FinProb}$ to a non-negative real number, so H_{BFL} can be regarded as a functor from $\mathsf{FinProb}$ to $\mathbb{BR}_+$. Roughly speaking, the main theorem can be said to claim that Segal entropy can be derived by adding Condition 3') to BFL's theorem.

Let us supplement on Condition 3) (or Condition 3')) of the main theorem. A pure-to-pure channel f maps pure states to pure states. Furthermore, it is easily seen that if f is also left-invertible, then it maps mutually orthogonal pure states to mutually orthogonal pure states. The latter half of Condition 3) claims that for such f_ω, the value of H is zero. As shown immediately below (see Eq. (6)), since H is a functor, we have $H(f_\omega) = S(\omega) - S(f \circ \omega)$. Here, if we regard $S(\omega)$ as the ambiguity possessed by the state ω, then $H(f_\omega)$ is the value obtained by subtracting the ambiguity possessed by the state $f \circ \omega$ from the ambiguity possessed by the state ω, i.e., it can be said to be a value representing how much ambiguity is reduced by the channel f. If ambiguity increases, then $H(f_\omega) < 0$ holds. Condition 3) claims that any pure-to-pure channel does not increase such ambiguity, and that f preserves ambiguity if it is also left-invertible. Equation (5) means that $S(\omega)$ is expressed in the form of $c\mathrm{S}_{\mathrm{Se}}(\omega)$.

IV. Proof of the main theorem

We will now prove the main theorem. Instead of Conditions 2) and 3), we will use Conditions 2') and 3'). In the proof, we will use BFL's theorem [1]. Note that BFL's theorem is based on the result of Ref. [8].

Since any channel $f \in \mathsf{Chn}(A, B)$ satisfies $\mathrm{Tr}^B \circ f = \mathrm{Tr}^A$, we have that for any $\omega \in \mathsf{St}_A$,

$$H(\mathrm{Tr}^B_{f\circ\omega}) + H(f_\omega) = H(\mathrm{Tr}^B_{f\circ\omega} \circ f_\omega) = H(\mathrm{Tr}^A_\omega),$$

where the first equality follows from the functoriality of H, and the second equality follows from the fact that $\mathrm{Tr}^B_{f\circ\omega} \circ f_\omega = \mathrm{Tr}^A_\omega$, which is obtained from Eq. (3). Therefore, from Eq. (4), we have

$$H(f_\omega) = S(\omega) - S(f \circ \omega). \tag{6}$$

Let $H_{\mathrm{BFL}} := HI$, where $I \colon \mathsf{FinProb} \to \mathsf{FinState}$ is the inclusion functor. Then, as already mentioned, from Conditions 1) and 2') of Theorem 3, Conditions 1) and 2) of BFL's theorem are obtained. Also, from condition 3') of Theorem 3, it is understood that H_{BFL} is a functor from $\mathsf{FinProb}$ to $\mathbb{BR}_+$. Therefore, from BFL's theorem, we obtain Eq. (2). That is, there exists a non-negative real number c such that, for any pure-to-pure channel $f \in \mathsf{Chn}(\mathbb{C}^X, \mathbb{C}^Y)$ and $p \in \mathsf{St}_{\mathbb{C}^X}$,

$$H(f_p) = c(\mathrm{S}_{\mathrm{Sh}}(p) - \mathrm{S}_{\mathrm{Sh}}(f \circ p))$$

holds. In particular, considering the case of $f = \mathrm{Tr}^{\mathbb{C}^X}$, we obtain

$$S(p) = c\mathrm{S}_{\mathrm{Sh}}(p), \quad \forall p \in \mathsf{St}_{\mathbb{C}^X}, \tag{7}$$

where we use $S(p) = H(\mathrm{Tr}^{\mathbb{C}^X}_p)$ and $\mathrm{S}_{\mathrm{Sh}}(\mathrm{Tr}^{\mathbb{C}^X} \circ p) = \mathrm{S}_{\mathrm{Sh}}(1) = 0$.

Let us arbitrarily choose a system A and its state $\omega \in \mathsf{St}_A$ and express A in the form of Eq. (1). Then, there exists a set of orthogonal pure states $\{\psi_i \in \mathsf{St}_A\}_{i=1}^n$ with $n := \sum_{i=1}^k n_i$ such that ω is represented by

$$\omega = \sum_{i=1}^n \gamma_i \psi_i, \quad \gamma_i \geq 0, \quad \sum_{i=1}^n \gamma_i = 1. \tag{8}$$

In this case, we have

$$\mathrm{S}_{\mathrm{Se}}(\omega) = -\sum_{i=1}^n \gamma_i \log \gamma_i. \tag{9}$$

Consider the channel

$$f \colon \mathsf{St}_{\mathbb{C}^Z} \ni p \mapsto \sum_{i=1}^n \mathrm{Tr}(\phi_i^Z p) \cdot \psi_i \in \mathsf{St}_A$$

from the classical system $\mathbb{C}^Z$ with $Z := \{1, 2, \ldots, n\}$ to A. Since f maps each pure state ϕ_i^Z ($i \in \{1, \ldots, n\}$) of $\mathbb{C}^Z$ to ψ_i, it is pure-to-pure. Also, consider the channel

$$g\colon \mathsf{St}_A \ni \omega \mapsto \sum_{i=1}^{n} \mathrm{Tr}(\psi_i \omega) \cdot \phi_i^Z \in \mathsf{St}_{\mathbb{C}^Z}$$

from A to $\mathbb{C}^Z$. Then, it can be seen that $g \circ f$ is the identity channel on $\mathbb{C}^Z$. Therefore, f is left-invertible. For ω expressed by Eq. (8), let $\gamma := g \circ \omega = \sum_{i=1}^{n} \gamma_i \phi_i^Z$. Since $f \circ \gamma = \omega$, we obtain

$$H(f_\gamma) = S(\gamma) - S(\omega) = c\mathrm{S}_{\mathrm{Sh}}(\gamma) - S(\omega),$$

where the first and second equalities follow from Eqs. (6) and (7), respectively. On the other hand, since $H(f_\gamma) = 0$ holds from Condition 3'), we have

$$S(\omega) = c\mathrm{S}_{\mathrm{Sh}}(\gamma) = -c \sum_{i=1}^{n} \gamma_i \log \gamma_i = c\mathrm{S}_{\mathrm{Se}}(\omega),$$

where the last equality follows from Eq. (9). Substituting this into Eq. (6), we obtain Eq. (5), which completes the proof of the main theorem.

V. Conclusion

We have proposed a method to extend BFL's theorem to quantum systems and characterize Segal entropy (or von Neumann entropy). Specifically, we showed that if a functor from $\mathsf{FinState}$ to $\mathbb{BR}$ satisfies certain properties, it can be expressed as a constant multiple of the difference in Segal entropy.

Acknowledgment

I am grateful to O. Hirota, M. Sohma, and K. Kato for support. This work was supported by the Air Force Office of Scientific Research under award number FA2386-22-1-4056.

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