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Cut-off Rate for ASK signal states

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***Abstract*—We compute a cut-off rate R_M for M-ary ASK signal states and deal with the discretization problem where we consider whether R_M achieves a continuous cut-off rate.**

I. Introduction

This paper discusses a cut-off rate of quantum Gaussian channels, where classical information is conveyed by quantum Gaussian states and a positive operator valued measure is used in the decoding procedure. In particular we mainly deal with ASK signal states.

The quantum cut-off rate is a quantity describing behavior of error probability exponents at medium rates[1]. The Holevo capacity gives the upper limit on the information rate for reliable classical-quantum communication and the quantum cut-off rate is considered to be the practical upper bound on the information rate of a classical-quantum channel[2]. Ban et.al developed a method of computing quantum cut-off rate and applied it for a group-covariant M-ary quantum state channel[2].

Our purpose is to reveal properties of ASK signal states in terms of quantum cut-off rate. Unfortunately, for ASK signal states, we cannot always apply the Ban's method of computing a quantum cut-off rate. So we compute it by exploiting a numerical method in Sec. II. In Sec. III we obtain a continuous quantum cut-off rate in the case of one dimensional distributed signals and compare it with the result for the discrete case.

II. Quantum Cut-off Rate for ASK signal states

We firstly remind the quantum cut-off rate for classical-quantum communication channels with M pure signal states $\{|\psi_1\rangle, ..., |\psi_M\rangle\}$. It is given by

$$R_M = \max_{\pi} \tilde{\mu}(\pi, 1) \tag{1}$$

where the function $\tilde{\mu}(\pi, s)$ is a Gallager function given as

$$\tilde{\mu}(\pi, s) = -s \ln \sum_{j=1}^{M} \sum_{k=1}^{M} \pi_j \pi_k |\langle \psi_j | \psi_k \rangle|^{2/s}. \tag{2}$$

Ban found that the quantum cut-off rate can be computed as

$$R_M = \ln \left[\sum_{j=1}^{M} \sum_{k=1}^{M} (\mathcal{G}_2^{-1})_{jk} \right], \tag{3}$$

if

$$\tilde{\pi}_j = \frac{\sum_{k=1}^{M} (\mathcal{G}_2^{-1})_{jk}}{\sum_{i=1}^{M} \sum_{k=1}^{M} (\mathcal{G}_2^{-1})_{ik}}, \tag{4}$$

is non-negative for all $j = 1, 2, ..., M$, where $\mathcal{G}_2^{-1}$ is the inverse of the matrix $(\mathcal{G}_2)_{jk} = |\langle \psi_j | \psi_k \rangle|^2$ [2]. Note that $\{\tilde{\pi}_j\}$ gives the optimum input probability when the above condition is satisfied.

Let us compute the quantum cut-off rate for ASK signal states, which consists of M signal states $\{|-\alpha\rangle, ..., |\alpha\rangle\}$. Here we assume α is a real number for simplicity. Unfortunately we cannot always employ Ban's formula (3) because $\tilde{\pi}_j$ may not be positive when distance between signals is short. Then we must rely on numerical computation. Fig. 1 shows graphs of cut-off rates R_M with respect to number of signals M for $\alpha = 2, 5, 10$. In Fig. 1 circles are computed by Ban's formula. The graphs indicate that large number of signals is needless. Unlike the case of PSK signal states the average energy

$$N = \sum_{j=1}^{M} \pi_j |\alpha_j|^2 \tag{5}$$

with $\{|\alpha_1\rangle, ..., |\alpha_M\rangle\} = \{|-\alpha\rangle, ..., |\alpha\rangle\}$, changes as *a priori* probability distribution π does. We are interested in knowing how the average energy, N_M, for the optimum *a priori* distribution changes according to the number of signals, M. Fig. 2 shows the graph of N_M/N_2, with respect to number of signals M for $\alpha = 2, 5, 10$. Here we use a normalization N_M/N_2 instead of N_M, because we are interested in whether we need a larger energy when M takes a larger value.

III. Discretization

We remind the quantum cut-off rate for a continuous classical-quantum channel with pure signal states $\{|\psi_m\rangle; m \in \mathcal{M}\}$ where $\mathcal{M}$ is a Borel subset in a finite dimensional Euclidean space. In [6] it is given as

$$R_C = \max_{0 \le p} \max_{\pi \in \mathcal{P}_1} \tilde{\mu}(\pi, 1, p), \tag{6}$$

where $\mathcal{P}_1$ is the set of probability distribution π satisfying $\int f(m)\pi(dm) \le E$ for a fixed nonnegative Borel function f on $\mathcal{M}$ and

$$\tilde{\mu}(\pi, s, p) = -s \ln \int \mathrm{e}^{p[f(m)+f(n)-2E]} |\langle \psi_m | \psi_n \rangle|^{2/s} \pi(dm)\pi(dn). \tag{7}$$

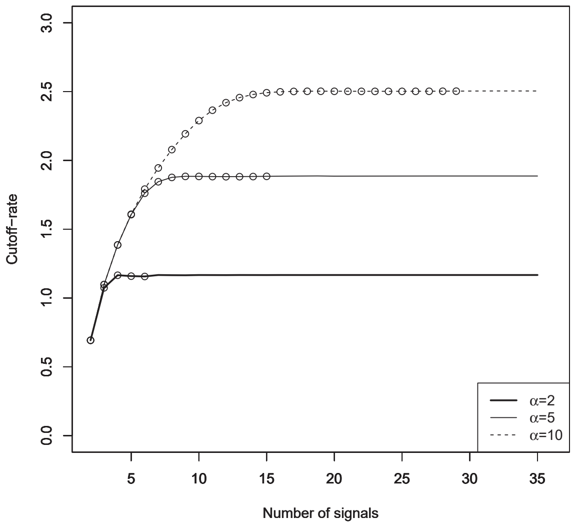

Fig. 1. Dependence of cut-off rate on number of signals, M, when $\alpha = 2, 5, 10$. Circles show values computed by Ban's formula.

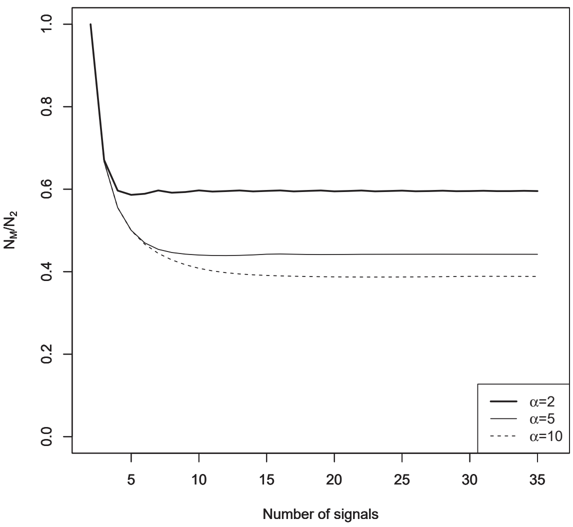

Fig. 2. Dependence of N_M/N_2 on number of signals M, when $\alpha = 2, 5, 10$.

This is the quantum cut-off rate that we can achieve if we are allowed to use codes $|\psi_{m_1}\rangle \otimes \cdots \otimes |\psi_{m_K}\rangle$ satisfying energy constraint

$$f(m_1) + \cdots + f(m_K) \leq KE. \tag{8}$$

On the other hand in the case of quantum cut-off rate R_M any letter states $|\psi_{m_j}\rangle$ in a codeword are chosen from the fixed finite set $\{|\psi_1\rangle, ..., |\psi_M\rangle\}$ and energy constraint is not considered. Putting $p = 0$ and considering a discrete probability distribution as π in Eq. (6), we obtain the following relation

$$R_M \leq R_C, \tag{9}$$

where $\{|\psi_1\rangle, ..., |\psi_M\rangle\} \subset \{|\psi_m\rangle; m \in \mathcal{M}\}$ and energy constraint E is fixed to the value of average energy with optimum probability distribution in Eq. (1).

In the following we devote ourselves to the case of coherent signal states. Then we consider

$$f(\alpha) = \hbar|\alpha|^2 \tag{10}$$

as a signal energy for coherent state $|\alpha\rangle$ and put $E = \hbar N_{tr}$.

Let us compute the Gallager function assuming *a priori* probability distribution is Gaussian

$$\pi(d^2\alpha) = \frac{1}{\pi N_{tr}} \exp\left(-\frac{|\alpha|^2}{N_{tr}}\right) d^2\alpha, \tag{11}$$

where α is a complex valued random variable. Then we have [6]

$$\tilde{\mu}(\pi, s, p) = s\left[2pE + \log\left(1 + p^2E^2 - 2pE + \frac{E(1-pE)}{\hbar s}\right)\right], \tag{12}$$

with $E = \hbar N_{tr}$, and we can solve the optimization (6) and obtain

$$R_C = 2N_{tr} + 2 - 2\vartheta(2N_{tr}) + \ln \vartheta(2N_{tr}), \tag{13}$$

with $\vartheta(t) = (1 + \sqrt{t^2+1})/2$. It is more suitable to consider the case where *a priori* probability π is distributed one-dimensionally:

$$\pi(dx) = \frac{1}{\sqrt{2\pi N_{tr}}} \exp\left(-\frac{|x|^2}{2N_{tr}}\right) dx. \tag{14}$$

Then we compute the cut-off rate R_x as

$$R_x = 2N_{tr} + 1 - \vartheta(4N_{tr}) + \frac{\ln \vartheta(4N_{tr})}{2}. \tag{15}$$

Here we have the relation $R_M \leq R_x \leq R_C$ and R_M ($M = 2, 3, ...$) increases monotonously with respect to M. Fig. 3 shows the graphs of R_2/R_x and R_{30}/R_x with respect to average signal energy N_{tr}. Note that we have $N_M \leq N_{30}$ for $M < 30$ and $N_M = N_{30}$ for $M > 30$ in our case, where N_{tr} is small ($N_{tr} < 0.8$). These graphs show the followings.

1) When the average energy N_{tr} is small ($N_{tr} < 0.1$), the binary cut-off rate R_2 has almost the same value as the continuous cut-off rate R_x.
2) When $N_{tr} < 0.6$, it is not necessary to use more than two signal states. i.e. $N_M = N_2, M = 3, 4, ..$
3) When N_{tr} has the large value, the value of R_M does not approach to that of R_x even if we take any large values of M.

By further computation we can find graphs of R_M/R_x, $M \geq 3$ coincide each other when N_{tr} is small (e.g. $N_{tr} \leq$

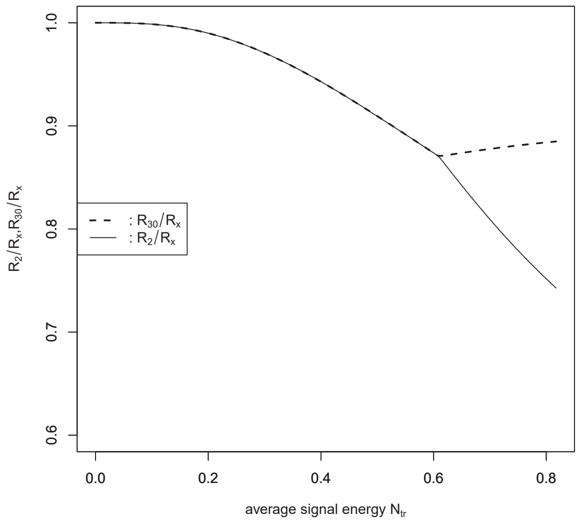

Fig. 3. Dependence of ratios R_2/R_x,R_{30}/R_x on average signal energy, N_{tr}.

0.8). So the dot line in Fig. 3 gives an upper bound. On the other hand for a larger value of N_{tr} we need a larger value of M in order to achieve the upper bound.

IV. Conclusion

We have computed the cut-off rate R_M for M-ary ASK signal states $\{|-\alpha\rangle, ..., |\alpha\rangle\}$ and compared it with the continuous cut-off rate R_x. The value of binary cut-off rate R_2 is equal to that of R_x approximately when $N_{tr} << 1$ while R_M does not achieve the continuous cut-off rate even if we take any large values of M. This means that our strategy based on ASK signal states is not suitable to achieve the continuous cut-off rate for a large value of N_{tr}.

References

[1] M. V. Burnashev and A. S. Holevo, On reliability function of quantum communication channel, Probl. Peredachi Inform., 34, 1-13, 1998

[2] M. Ban and K. Kurokawa and O. Hirota Cut-off rate for quantum communication channels with entangled measurement, Quantum Semiclass. Opt., 10, L7-L12, 1998

[3] Fumio Futami, Experimental demonstrations of Y-00 cipher for high capacity and secure optical fiber communications, Quantum Information Processing, vol. 13, no-10, pp2277-2292, 2014

[4] A. S. Holevo, Coding theorems for quantum channels, Tamagawa University Research Review, 4, 1, 1998

[5] A. S. Holevo, Reliability function of general classical-quantum channel, IEEE Trans. on Information Theory, 46(6), 2256-2260, 2000

[6] A. S. Holevo and M. Sohma and O. Hirota, Error exponents for quantum channels with constrained inputs,Reports on Mathematical Physics, 46, 3, 343-358, 2000

[7] Y. Ishida, K. Kato, and T. S. Usuda, Capacity of attenuated channel with discrete-valued input, Proceedings of the 8th International Conference on Quantum Communication, Measurement and Computing, O. Hirota, J. H. Shapiro, and M. Sasaki (Eds.), NICT Press, pp.323-326, 2007.