ISSN 2186-6570

Remarks on Some Results on Rotations

Mitsuru Hamada

Quantum Information Science Research Center
Quantum ICT Research Institute
Tamagawa University
6-1-1 Tamagawa-gakuen, Machida, Tokyo 194-8610, Japan

Tamagawa University Quantum ICT Research Institute Bulletin, Vol.6, No.1, 1-4, 2016

Tamagawa University Quantum ICT Research Institute Bulletin
Vol.6 No.1 : 1–4 (2016)

Remarks on Some Results on Rotations

Mitsuru Hamada

Quantum Information Science Research Center
Quantum ICT Research Institute
Tamagawa University
6-1-1 Tamagawa-gakuen, Machida, Tokyo 194-8610, Japan

I. Introduction

Motivated by some problems on quantum computation, the present author has recently investigated issues on rotations in the Euclidean space or the corresponding unitary operations [1], [2], [3]. Specifically, in a previous article [2] in this bulletin, the author described a result of an unpublished manuscript on rotations (its abstract is available online [1]). As the title suggests, it contains some fundamental lemma on Euler angles. The lemma has already led to a result on construction of unitary operations or rotations [3]. The original lemma was reduced to a neat form, which the present author thought was natural when it was included in [3]. Therefore, the published form of the lemma in [3] is not exactly the same as the original form described in [1], [2]. In this memorandum, a relation between these two forms of the lemma will be described. Specifically, they will be shown to be essentially equivalent to each other based on some basic facts on rotations.

The primary reason for presenting the proof of equivalence is that the equivalence may not be immediate to see unless one is familiar with some basic properties of rotations, and such basics are likely to be forgotten since the set of rotations is typically regarded as a (relatively easy) special case of general mathematical objects such as the Lie groups. [Besides, we remind the reader of the fact that, as several authors have remarked in the literature (see the previous articles [2], [4] in this bulletin), around issues on SU(2) and SO(3), whereas these are fundamental in physics, quite many errors can be found in the literature. In this author's opinion, one source of these errors would be the lack of acquirement of the logical (and rigorous) way of thinking. Below, from this viewpoint, he will offer a proof (of this annotative statement) that may be persistently more logical than usual.]

In addition, a constructive result on rotations is described. Namely, a constructive method (proof) for obtaining the inverse image of an arbitrary element in SO(3) under the well-known homomorphism from SU(2) onto SO(3) is presented. (In this article, the very basic knowledges on algebra such as homomorphisms and kernels are assumed.)

II. Definitions

Let X, Y and Z denote the Pauli matrices:

$$X = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad Y = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad Z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

Throughout, I denotes the 2×2 identity matrix. The transpose of a vector $\hat{v}$ is denoted by $\hat{v}^{\mathrm{T}}$.

We put

$$R_{\hat{v}}(\theta) = (\cos\tfrac{\theta}{2})I - i(\sin\tfrac{\theta}{2})(v_x X + v_y Y + v_z Z) \qquad (1)$$

where $\hat{v} = (v_x, v_y, v_z)^{\mathrm{T}} \in \mathbb{R}^3$ with $\|\hat{v}\| = \sqrt{v_x^2 + v_y^2 + v_z^2} = 1$ and $\theta \in \mathbb{R}$, with $\mathbb{R}$ denoting the set of real numbers. The matrices $R_y(\theta)$ and $R_z(\theta)$ denote the following special cases of $R_{\hat{v}}$, respectively: $R_y(\theta) = R_{\hat{y}}(\theta)$, where $\hat{y} = (0, 1, 0)^{\mathrm{T}}$, and $R_z(\theta) = R_{\hat{z}}(\theta)$, where $\hat{z} = (0, 0, 1)^{\mathrm{T}}$.

We put $S^2 = \{\hat{v} \in \mathbb{R}^3 \mid \|v\| = 1\}$. The set of 2×2 unitary matrices, the set of 2×2 unitary matrices with determinant 1 and the set of 3×3 real orthogonal matrices with determinant 1 are denoted by U(2), SU(2) and SO(3), respectively. They stand for the unitary group, the special unitary group and the special orthogonal group, respectively.

III. The Fundamental Lemma on Euler Angles in Several Forms

We have the following lemma.

Lemma 1: [1, Theorem]. For any $\phi, \beta, \theta \in \mathbb{R}$ and $\hat{v} = (v_x, v_y, v_z)^{\mathrm{T}} \in \mathbb{R}^3$ with $v_x^2 + v_y^2 + v_z^2 = 1$, the following two conditions are equivalent.

I. There exist some $\alpha, \gamma \in \mathbb{R}$ such that

$$R_{\hat{v}}(\theta) = e^{i\phi} R_z(\alpha) R_y(\beta) R_z(\gamma).$$

II. Both of the following hold:

$$e^{i\phi} \in \{1, -1\},$$
$$\sqrt{1 - v_z^2}\,|\sin\tfrac{\theta}{2}| = |\sin\tfrac{\beta}{2}|.$$

The original manuscript the main result of which is this lemma is unpublished. That manuscript also includes the following (trivially equivalent but useful) form of the above lemma.

Lemma 2: For any $\phi, \beta, \theta \in \mathbb{R}$ and for any $\hat{n}, \hat{l}, \hat{m} \in S^2$ such that $\hat{l}^{\mathrm{T}}\hat{m} = 0$, the following two conditions are equivalent.

I°. There exist some $\alpha, \gamma \in \mathbb{R}$ such that

$$R_{\hat{n}}(\theta) = e^{i\phi} R_{\hat{m}}(\alpha) R_{\hat{l}}(\beta) R_{\hat{m}}(\gamma).$$

II°. Both of the following hold:

$$e^{i\phi} \in \{1, -1\},$$
$$\sqrt{1-(\hat{m}^{\mathrm{T}}\hat{n})^2}|\sin\tfrac{\theta}{2}| = |\sin\tfrac{\beta}{2}|.$$

On the other hand, the author has stated that the following lemma is fundamental to the results in the recent work of this author [3].

Lemma 3: For any $\beta, \theta \in \mathbb{R}$ and for any $\hat{n}, \hat{l}, \hat{m} \in S^2$ such that $\hat{l}^{\mathrm{T}}\hat{m} = 0$, the following two conditions are equivalent.

I. There exist some $\alpha, \gamma \in \mathbb{R}$ such that

$$R_{\hat{n}}(\theta) = R_{\hat{m}}(\alpha)R_{\hat{l}}(\beta)R_{\hat{m}}(\gamma).$$

II. $\sqrt{1-(\hat{m}^{\mathrm{T}}\hat{n})^2}|\sin\frac{\theta}{2}| = |\sin\frac{\beta}{2}|$.

Ostensibly, Lemma 2 seems more detailed than Lemma 3. In fact, Lemma 2, if we set $\phi = 0$, readily implies Lemma 3.

In this article, we shall show that Lemma 3, with some basic facts, implies Lemma 2.

IV. Some Basics on Rotations

A. Rotations as SU(2) Elements

The following fact is well-known.

Property 1: For $\hat{v} = (v_x, v_y, v_z)^{\mathrm{T}} \in \mathbb{R}^3$ with $\|\hat{v}\| = 1$ and $\theta \in \mathbb{R}$,

$$R_{\hat{v}}(\theta)$$

lies in $\mathrm{SU}(2)$.

Proof. Put $w = \cos(\theta/2)$, $x = -v_x \sin(\theta/2), y = -v_y \sin(\theta/2)$ and $z = -v_z \sin(\theta/2)$. Then, we have $w^2 + x^2 + y^2 + z^2 = 1$, so that

$$R_{\hat{v}}(\theta) = wI + i(xX + yY + zZ) = \begin{pmatrix} w+iz & y+ix \\ -y+ix & w-iz \end{pmatrix} \tag{2}$$

is an element in $\mathrm{SU}(2)$. □

Conversely, any element in $\mathrm{SU}(2)$ can be written as $R_{\hat{v}}(\theta)$ for some $\hat{v}$ and θ (see Appendix if a proof is needed).

B. A relation of SU(2) to SO(3)

We define a map $\overline{F} : \mathrm{U}(2) \to \mathrm{SO}(3)$ as follows. With any matrix $U \in \mathrm{U}(2)$, we associate a 3×3 real matrix $R = \overline{F}(U)$ that satisfies

$$UM(x, y, z)U^{\dagger} = M(x', y', z')$$

and

$$\begin{pmatrix} x' \\ y' \\ z' \end{pmatrix} = R \begin{pmatrix} x \\ y \\ z \end{pmatrix}$$

for any $(x, y, z)^{\mathrm{T}} \in \mathbb{R}^3$.

The restriction of the homomorphism $\overline{F}$ to $\mathrm{SU}(2)$ is well-known [5]. This restriction is denoted by F.

Example. We have

$$\hat{R}_y(\theta) := F(R_y(\theta)) = \begin{pmatrix} \cos\theta & 0 & \sin\theta \\ 0 & 1 & 0 \\ -\sin\theta & 0 & \cos\theta \end{pmatrix} \tag{3}$$

and

$$\hat{R}_z(\theta) := F(R_z(\theta)) = \begin{pmatrix} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{pmatrix}. \tag{4}$$

Definition 1: For $\hat{v} = (v_x, v_y, v_z)^{\mathrm{T}} \in \mathbb{R}^3$ with $\|\hat{v}\| = 1$ and $\theta \in \mathbb{R}$,

$$\hat{R}_{\hat{v}}(\theta) = F(R_{\hat{v}}(\theta)).$$

One can check that through this homomorphism F, the matrix $R_{\hat{v}}(\theta)$ really acts as rotation about $\hat{v}$ by angle θ on $\mathbb{R}^3$ (see, e.g., [2, Section III]).

We shall also use the following fact.

Property 2: For any $U, V \in \mathrm{SU}(2)$, $F(U) = F(V)$ if and only if $U = \pm V$.

Proof. This directly follows from the well-known fact that the kernel of $F : \mathrm{SU}(2) \to \mathrm{SO}(3)$ is $\{I, -I\}$, which can be checked with (1). □

V. Proof of the Equivalence of Lemma 3 to Lemma 2

As already mentioned, Lemma 2 immediately implies Lemma 3. We shall show the converse using the basic facts given in Section IV.

First, we strengthen Lemma 3 slightly.

Lemma 4: For any $\beta, \theta \in \mathbb{R}$ and for any $\hat{n}, \hat{l}, \hat{m} \in S^2$ such that $\hat{l}^{\mathrm{T}}\hat{m} = 0$, the following three conditions are equivalent.

I. There exist some $\alpha, \gamma \in \mathbb{R}$ such that

$$R_{\hat{n}}(\theta) = R_{\hat{m}}(\alpha)R_{\hat{l}}(\beta)R_{\hat{m}}(\gamma).$$

Î. There exist some $\alpha, \gamma \in \mathbb{R}$ such that

$$\hat{R}_{\hat{n}}(\theta) = \hat{R}_{\hat{m}}(\alpha)\hat{R}_{\hat{l}}(\beta)\hat{R}_{\hat{m}}(\gamma).$$

II. $\sqrt{1-(\hat{m}^{\mathrm{T}}\hat{n})^2}|\sin\frac{\theta}{2}| = |\sin\frac{\beta}{2}|$.

Proof. It suffices to show that I $\leftrightarrow$ Î (for any $\beta, \theta \in \mathbb{R}$ and for any $\hat{n}, \hat{l}, \hat{m} \in S^2$ with $\hat{l}^{\mathrm{T}}\hat{m} = 0$). We immediately see the part [I $\to$ Î] applying the homomorphism F to both sides of the equation in I. Conversely, assume that Î holds. Then, by Property 2, there exist some $\alpha, \gamma \in \mathbb{R}$ such that

$$R_{\hat{n}}(\theta) = \pm R_{\hat{m}}(\alpha)R_{\hat{l}}(\beta)R_{\hat{m}}(\gamma).$$

But $-R_{\hat{m}}(\alpha) = R_{\hat{m}}(\alpha + 2\pi)$. Thus, we have condition I, and hence, the lemma. □

Up to now, we have shown implications

$$\text{Lemma 2} \to \text{Lemma 3} \to \text{Lemma 4}.$$

We shall show

$$\text{Lemma 4} \to \text{Lemma 2}.$$

To derive Lemma 2 from Lemma 4, assume that I° in Lemma 2 holds. Then, applying the homomorphism $\overline{F}$ to both sides of the equation in I°, we have Î and hence, II, which is a half of II°. To see the other half, we shall use Property 2. Note I° in Lemma 2, which is being assumed to hold, can be rewritten as

$$R_{\hat{n}}(\theta)[R_{\hat{m}}(\alpha)R_{\hat{l}}(\beta)R_{\hat{m}}(\gamma)]^{-1} = e^{i\phi}I.$$

Then, since $R_{\hat{n}}(\theta)$ and $[R_{\hat{m}}(\alpha)R_{\hat{l}}(\beta)R_{\hat{m}}(\gamma)]^{-1}$ lie in $\mathrm{SU}(2)$, $e^{i\phi}I$ must also lie in $\mathrm{SU}(2)$. Hence, we have $e^{i\phi} \in \{1, -1\}$, and II°.

Conversely, assume II°. Then, if $e^{i\phi} = 1$, by the part [II $\to$ I] of Lemma 4, we have I°; if $e^{i\phi} = -1$, using the relation $R_{\hat{m}}(\alpha) = -R_{\hat{m}}(\alpha + 2\pi)$, we have I° similarly.

Thus, we have shown the implications

Lemma 2 $\to$ Lemma 3 $\to$ Lemma 4 $\to$ Lemma 2.

This completes the proof of the equivalence among Lemma 3, Lemma 2 and Lemma 4.

VI. Inverse Image Under the Homomorphism from $\mathrm{SU}(2)$ onto $\mathrm{SO}(3)$

It is known that the map $F : \mathrm{SU}(2) \to \mathrm{SO}(3)$ is surjective (onto), i.e., that for any $R \in \mathrm{SO}(3)$, we have some $U \in \mathrm{SU}(2)$ with $R = F(U)$.

In this section, we shall prove this fact in a constructive manner, where one should note that $\mathrm{SO}(3)$ can be expressed as

$$\{(\hat{l} \times \hat{m} \;\; \hat{l} \;\; \hat{m}) \mid \hat{l}, \hat{m} \in \mathbb{R}^3, \|\hat{l}\| = \|\hat{m}\| = 1, \hat{l}^{\mathrm{T}}\hat{m} = 0\}. \tag{5}$$

Proof that the map F in Section IV-B is onto $\mathrm{SO}(3)$. In view of the expression of $\mathrm{SO}(3)$ in (5), our goal is to prove that for any pair of vectors $\hat{l}, \hat{m} \in \mathbb{R}^3$ with $\|\hat{l}\| = \|\hat{m}\| = 1$ and $\hat{l}^{\mathrm{T}}\hat{m} = 0$, there exists some element in $\mathrm{SU}(2)$ such that $\hat{l} = F(U)(0,1,0)^{\mathrm{T}}$, $\hat{m} = F(U)(0,0,1)^{\mathrm{T}}$ and $\hat{l} \times \hat{m} = F(U)(1,0,0)^{\mathrm{T}}$. Expressing U, in terms of Euler angles (Appendix), as

$$U = U_{\alpha,\beta,\gamma} := R_z(\alpha)R_y(\beta)R_z(\gamma),$$

we can calculate $F(U)$ directly (using Example in Section IV-B) as

$$F(U) = F(U_{\alpha,\beta,\gamma}) = \hat{R}_z(\alpha)\hat{R}_y(\beta)\hat{R}_z(\gamma)$$
$$= \begin{pmatrix} a & -\cos\alpha\cos\beta\sin\gamma - \sin\alpha\cos\gamma & \cos\alpha\sin\beta \\ b & -\sin\alpha\cos\beta\sin\gamma + \cos\alpha\cos\gamma & \sin\alpha\sin\beta \\ c & \sin\beta\sin\gamma & \cos\beta \end{pmatrix} \tag{6}$$

where $(a,b,c)^{\mathrm{T}}$ is the vector product of the second and third columns of $F(U)$. Moreover, the condition $\hat{l} = F(U)(0,1,0)^{\mathrm{T}}$ is equivalent to

$$\hat{R}_y(-\beta)\hat{R}_z(-\alpha)\hat{l} = \hat{R}_z(\gamma)(0,1,0)^{\mathrm{T}},$$

i.e.,

$$\begin{pmatrix} \cos\beta\cos\alpha & \cos\beta\sin\alpha & -\sin\beta \\ -\sin\alpha & \cos\alpha & 0 \\ \cos\alpha\sin\beta & \sin\alpha\sin\beta & \cos\beta \end{pmatrix} \hat{l} = \begin{pmatrix} -\sin\gamma \\ \cos\gamma \\ 0 \end{pmatrix}. \tag{7}$$

From (6) and (7), we conclude that for any pair of orthogonal unit vectors $\hat{l}$ and $\hat{m}$, there exists some $\alpha, \beta, \gamma \in \mathbb{R}$, and hence, an element $U = U_{\alpha,\beta,\gamma}$ in $\mathrm{SU}(2)$ such that $\hat{l} = F(U)(0,1,0)^{\mathrm{T}}$, $\hat{m} = F(U)(0,0,1)^{\mathrm{T}}$ and $\hat{l} \times \hat{m} = F(U)(1,0,0)^{\mathrm{T}}$, as desired. □

Remark on constructiveness. Given a rotation matrix $R \in \mathrm{SO}(3)$, the elements $U \in \mathrm{SU}(2)$ such that $R = F(U)$ are directly specified by (6) and (7). Namely, the relation $\hat{m} = (\cos\alpha\sin\beta, \sin\alpha\sin\beta, \cos\beta)^{\mathrm{T}}$, which is from (6), specifies α and β, cf. spherical coordinates, and (7) specifies γ. Thus, this proof is constructive.

Note that there are two elements $U \in \mathrm{SU}(2)$ such that $R = F(U)$ for each $R \in \mathrm{SO}(3)$. This is a consequence of the fact that the kernel of F is $\{I, -I\}$. (If one gets an element $U \in \mathrm{SU}(2)$ such that $R = F(U)$, with the above method or another, then the other element U' such that $R = F(U')$ is $U' = -U$.)

Here is another remark. The argument in the above proof and remark is essentially the same as that in [3, Appendix A]. The intention there was to construct an element in $\mathrm{SU}(2)$ such that $\hat{l} = F(U)(0,1,0)^{\mathrm{T}}$ and $\hat{m} = F(U)(0,0,1)^{\mathrm{T}}$. For the present purpose of obtaining the inverse image, the above proof explicitly mentions the form $(\hat{l} \times \hat{m} \;\; \hat{l} \;\; \hat{m})$ of an $\mathrm{SO}(3)$ element. This is the only difference.

VII. Conclusion

A fundamental lemma of [3, Lemma 6.1] has a form different from the original one (2012, unpublished). This article has shown that these different forms of the lemma imply each other. A constructive method for realizing the inverse of $\tilde{F} : \mathrm{SU}(2)/\{I, -I\} \to \mathrm{SO}(3)$ has been presented, where $\tilde{F}$ is the isomorphism induced by the well-known homomorphism F from $\mathrm{SU}(2)$ onto $\mathrm{SO}(3)$, which has the kernel $\{I, -I\}$.

Acknowledgments

This work was supported by JSPS KAKENHI Grant number JP26247016.

Appendix

Parameterizations of the Elements in $\mathrm{SU}(2)$

It can be shown easily that any matrix in $\mathrm{SU}(2)$ can be written as [5]

$$\begin{pmatrix} a & b \\ -b^* & a^* \end{pmatrix} \tag{8}$$

with some complex numbers a and b such that $|a|^2 + |b|^2 = 1$. Hence, any matrix in $\mathrm{SU}(2)$ can be written as

$$\begin{pmatrix} w + iz & y + ix \\ -y + ix & w - iz \end{pmatrix} = wI + i(xX + yY + zZ) \tag{9}$$

with some real numbers x, y, z and w such that $w^2 + x^2 + y^2 + z^2 = 1$. Take a real number θ such that $\cos(\theta/2) = w$ and $\sin(\theta/2) = \sqrt{1 - w^2} = \sqrt{x^2 + y^2 + z^2}$; write x, y and z as $x = -v_x \sin(\theta/2), y = -v_y \sin(\theta/2)$ and $z = -v_z \sin(\theta/2)$, where $v_x, v_y, v_z \in \mathbb{R}$ and $v_x^2 + v_y^2 + v_z^2 = 1$.

Thus, using real numbers $\theta, v_x, v_y, v_z \in \mathbb{R}$ with $v_x^2 + v_y^2 + v_z^2 = 1$, any matrix in $\mathrm{SU}(2)$ can be written as

$$(\cos\tfrac{\theta}{2})I - i(\sin\tfrac{\theta}{2})(v_x X + v_y Y + v_z Z),$$

which is nothing but $R_{\hat{v}}(\theta)$ in (1).

A better-known parameterization for $\mathrm{SU}(2)$ would be

$$\begin{pmatrix} e^{-i\frac{\gamma+\alpha}{2}}\cos\frac{\beta}{2} & -e^{i\frac{\gamma-\alpha}{2}}\sin\frac{\beta}{2} \\ e^{-i\frac{\gamma-\alpha}{2}}\sin\frac{\beta}{2} & e^{i\frac{\gamma+\alpha}{2}}\cos\frac{\beta}{2} \end{pmatrix} = R_z(\alpha)R_y(\beta)R_z(\gamma). \tag{10}$$

Here, α, β and γ are real numbers, which are called Euler angles. This parameterization can be obtained by rewriting (8).

REFERENCES

[1] M. Hamada, "Overlooked restrictions on Euler angles in quantum computation," *APS 2013 March Meeting*, 2012, abstract, http://meetings.aps.org/link/BAPS.2013.MAR.H1.318.

[2] ——, "A lemma on Euler angles," *Tamagawa University Quantum ICT Research Institute Bulletin*, vol. 3, no. 1, pp. 25–27, Dec. 2013.

[3] ——, "The minimum number of rotations about two axes for constructing an arbitrarily fixed rotation," *Royal Society Open Science*, vol. 1, p. 140145, Nov. 2014, http://dx.doi.org/10.1098/rsos.140145.

[4] ——, "On parameterizations for rotations," *Tamagawa University Quantum ICT Research Institute Bulletin*, vol. 5, no. 1, pp. 25–28, Dec. 2015. Correction: on p. 26, right column, 12th and 14th lines from the bottom, the two colons should be deleted.

[5] E. P. Wigner, *Group Theory and Its Application to the Quantum Mechanics of Atomic Spectra*. New York: Academic Press, 1959.