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***Abstract*—In the 1980s, the present author discussed that a reduction of inner product between nonorthogonal states may be possible by means of quantum state control processes which are described by non unitary process. To discuss such processes, the author introduced an idea of conditional isometric operator as representation theory in 1994. Recently, Childs and Young have discovered a physical model for reduction of inner product of nonorthogonal states by nonlinear dynamics under the Gross-Pitaevskii and related nonlinearities. This paper clarifies a relation between conditional isometric operators and Childs-Young model, and discusses some limitations of their model in its application to general dynamic processes.**

I. Introduction

A reduction of quantum noise effect in quantum measurement process is a fundamental problem in quantum sciences. However, the present quantum theory imposes several fundamental limitations on the accuracy of the quantum measurement process. Such limitations come from nonorthogonality of quantum states. Almost all of the important theorems in quantum sciences based on the conventional quantum mechanics are related with such a nonorthogonality, which was formulated from 1960s to 1980s [1,2,3,4]. It is reviewed in my previous paper [5]. In contrast with such a development of quantum science, the author gave a prediction as follows [6]:
$\{Prediction\}$
There exists a dynamic process to reduce the inner product between two quantum states which is described by non unitary process.

Under this prediction, we challenged to discover physical model phenomenologically. However, almost all of such trials were failure or incomplete [7,8,9]. In general, the dynamics of quantum state in the conventional quantum mechanics obeys linear Schrödinger equation or evolution described by unitary operator. In other words, Schrödinger equation preserving inner product is in principle linear. In what follows, Morikawa-Takahara-Hirota [7] tried to verify the prediction by means of nonlinear quantum mechanics which was proposed by Weinberg [10,11], in which we employed an analysis based on nonlinear Schrödinger equation from quantum field theoretical point of view such as

$$i\frac{\partial\phi}{\partial t}+\frac{\partial^2\phi}{\partial x^2}-2\kappa|\phi|^2\phi=0 \tag{1}$$

where

$$[\phi(x,t),\phi^\dagger(y,t)]=\delta(x-y) \tag{2}$$
$$[\phi(x,t),\phi(y,t)]=[\phi^\dagger(x,t),\phi^\dagger(y,t)]=0 \tag{3}$$

However, the flow of the analysis included many assumptions and difficulties for calculus, and we could not obtain any clear result, except for the necessity of nonlinear quantum mechanics. To provide guidance for future work, the author gave mathematical concept for conditional quantum channel in 1994 [12].

Recently, Childs and Young have discovered a physical model for reduction of inner product between two nonorthogonal states by nonlinear dynamics under the Gross-Pitaevskii and related nonlinearities [13]. The author believes that their results will provide tremendously great progress in quantum science. Thus, the author reproduces the theory of conditional isometric operator in Section 2, and introduces Childs-Young theory on phenomenological process of nonlinear quantum mechanics in Section 3. In Section 4, a relation between conditional isometric operators and Childs-Young model is clarified. A universality of nonlinear quantum mechanic is discussed in Section 5, showing certain limitations of the applications of the nonlinear quantum mechanics to general problems.

II. Theory of conditional isometric operator

Physical process of a reduction of the inner product between two initial quantum states may correspond to a transformation of quantum state by some control processes, but it is not simply a unitary transformation. It was very difficult to discover a physical phenomenon to verify the hypothesis. So the author intended to give a description of such a model by operator theory and gave a mathematical definition so-called conditional isometric operator in 1994 [12]. Let us here describe the summary of the theory.

A. Mathematical definition

The nonlinear operator $\mathbf{A}$ on a subset $\mathcal{S} \subset \mathcal{H}$ of a Hilbert space $\mathcal{H}$ is isometric if

$$\|\mathbf{A}|\psi>\| = \||\psi>\| \tag{4}$$

When the range of an isometric operator is equal to its domain $\mathcal{S}$, it is called a nonlinear unitary operator. Linear isometric and unitary operators preserve the inner product for different elements of $\mathcal{H}$. Here I would like to eliminate, under certain conditions, the invariance of inner products. In order to do so, we must accept certain new ideas. In what follows, I present a conceptual definition. $\{Definition\}$
Let $\mathcal{S} : |\psi_j>, j \in J$ be a family of state vectors in $\mathcal{H}$, and let to each vector $|\psi_j>$ on this family correspond a linear operator $T_j = T(K_j)$ such that for $j \in J$

$$\|T_j|\psi_j>\| = \||\psi_j>\| \tag{5}$$

Then the nonlinear isometry $|\psi_j> \mapsto T_j|\psi_j>$ is called a conditional linear isometric operator on $\mathcal{S}$. K_j is initial state dependent Hamiltonian for the dynamics and the index $j \in J$ is called *susceptor* for the initial state. So it corresponds to initial state dependent dynamics.

In addition, when T_j is decomposed into linear combination: $T_j = \sum_m T^j_m$, T_j is defined as a conditional linear isometric operator such that

$$|\psi_j> \mapsto (\sum_m T^j_m)|\psi_j> = |\psi_j> \in \mathcal{S}_{\mathcal{Q}} \tag{6}$$

where $\mathcal{S}_{\mathcal{Q}}$ is quasi quantum state space in which the inner product is defined as follows:

$$<\psi_j|\psi_k> = \frac{\{<\psi_j|(\sum_m T^{\dagger j}_m)(\sum_n T^k_n)|\psi_k>\}\delta_{m,n}}{W_j \times W_k} \tag{7}$$

where

$$\begin{aligned} W_j \times W_k = \{<\psi_j|(\sum_m T^{\dagger j}_m)(\sum_n T^k_n)|\psi_j>\} \\ \delta_{m,n} \times \{<\psi_k|(\sum_m T^{\dagger j}_m)(\sum_n T^k_n)|\psi_k>\} \end{aligned} \tag{8}$$

and the projection form for the space in this state becomes

$$|\psi_j><\psi_j| = (\sum_m T^j_m)|\psi_j><\psi_j|(\sum_n T^{\dagger j}_n)\delta_{m,n} \tag{9}$$

T^j_m will be derived from a modification of mixed operation.

According to the above definition, one can represent transformation channels from pure state to pure state and also from pure state to mixed state as conditional isometric process.

Thus, one has a description method for quantum channel or dynamics with the following property:

$$<\psi_j|T^\dagger_j T_k|\psi_k> \neq <\psi_j|\psi_k> \quad j \neq k \tag{10}$$

Here one can classify such a nonlinear quantum control channel as follows:

(a) Positive conditional isometric channel=The absolute value of inner product increases.
(b) Negative conditional isometric channel=The absolute value of inner product decreases.

Of course the negative case is our goal. From the definition, T_j has *susceptor* j for initial state. So it corresponds to a reduction process of inner product by initial state dependent dynamics.

III. Childs-Young theory

Let us give a brief review of Childs-Young theory [13]. They employed the following nonlinear Schrödinger equation:

$$i\frac{d|\psi(t)>}{dt} = H(t)|\psi(t)> + K|\psi(t)> \tag{11}$$

where $H(t)$ is a time dependent Hermitian operator and K is a nonlinearity of the form

$$<x|(K|\psi>) = \kappa(|<x|\psi>|)<x|\psi> \tag{12}$$

and where $\kappa : [0,1] \to \mathcal{R}$ is a function characterizing the nonlinearity. In the Gross-Pitaevskii model, $\kappa(x) = gx^2$.

The Hermitian operator $H(t)$ gives the conventional time evolution. The main problem is to verify a reduction of inner product decreases on Bloch sphere.

They consider the pure state with Bloch sphere coordinate (x, y, z):

$$\rho = \frac{1}{2}\begin{bmatrix} 1+z & x-iy \\ x+iy & 1-z \end{bmatrix} \tag{13}$$

Since $|<0|\psi>|^2 = \frac{1+z}{2}$ and $|<1|\psi>|^2 = \frac{1-z}{2}$, the nonlinear term is clearly the state dependent Hamiltonian

$$\begin{bmatrix} \kappa((\frac{1+z}{2})^{1/2}) & 0 \\ 0 & \kappa((\frac{1-z}{2})^{1/2}) \end{bmatrix} \tag{14}$$

Then they found

$$\frac{d}{dt}(x, y, z) = \bar{\kappa}(z)(-y, x, 0) \tag{15}$$

where

$$\bar{\kappa} = \kappa((\frac{1+z}{2})^{1/2}) - \kappa((\frac{1-z}{2})^{1/2}) \tag{16}$$

Under these dynamics, states rotate around lines of latitude on the Bloch sphere.

Finally, Childs and Young give that the rate of change of the inner product of Bloch vectors (x_+, y_+, z_+) and (x_-, y_-, z_-), which are described by parameters α, ϕ, θ on Bloch sphere, is

$$\begin{aligned} & \frac{d}{dt}(x_+x_-, y_+y_-, z_+z_-) \\ = \; & (x_+y_- - y_+x_-)(\bar{\kappa}(z_+) - \bar{\kappa}(z_-)) \end{aligned} \tag{17}$$

When the angle of two states at the initial point is α_0 on the Bloch sphere, the rate of change of the inner product $\cos\alpha$ is given as follows:

$$\frac{d}{dt}\cos\alpha(t) = \sin\alpha(t)\sin\phi\sin\theta(\bar{\kappa}(z_+) - \bar{\kappa}(z_-)) \quad (18)$$

They showed that the above equation may provide the reduction of the inner product. So the author would like to interpret their result by the conditional isometric operator of the section 2 in the following.

IV. Relation with conditional isometric operator

In the Childs-Young theory, K plays an essential role, and it is equivalent to the initial state dependent Hamiltonian. Let us discuss how to connect the relation between differential equation representation and operator representation in the following.

K is related with $susceptor = \{j, k\}$ as the initial state dependency of the conditional isometric operator. That is, we have

$$i\frac{d|\psi_j(t)>}{dt} = (H(t) + K_j)|\psi_j(t)> \quad (19)$$

This equation provides dynamics with initial state dependency. In the operator representation, the solution of the differential equation corresponds to

$$\begin{aligned} |\bar{\psi}_j(t)> &= T(H(t) + K_j)|\psi_j> \\ &\equiv T_j(t)|\psi_j> \end{aligned} \quad (20)$$

where $|\psi_j>$ is the initial state with index j which means initial state dependency. And also one has

$$i\frac{d|\psi_k(t)>}{dt} = (H(t) + K_k)|\psi_k(t)> \quad (21)$$

$$\begin{aligned} |\bar{\psi}_k(t)> &= T(H(t) + K_k)|\psi_k> \\ &\equiv T_k(t)|\psi_k> \end{aligned} \quad (22)$$

where k is initial state index. If one has

$$\mathbf{d} =< \psi_j|T_j^\dagger T_k|\psi_k> \quad < \quad \mathbf{d}_0 =< \psi_j|\psi_k> \quad j \neq k \quad (23)$$

this is, in fact, the negative conditional isometric process.

Finally, the rate equation (Eq-18) of changing of the inner product by Childs-Young means $\cos\alpha(t) = \mathbf{d}(t)$. Since they employ the Gross-Pitaevskii nonlinearity, they have $\kappa(x) = gx^2$, and $\bar{\kappa}(z) = gz$. Thus the test of the negative conditional isometric process is given by their equation:

$$\begin{aligned} \frac{d}{dt}\mathbf{d}(t) &= \frac{d}{dt}\cos\alpha(t) \\ &= g\sin(\alpha)\sin(\alpha/2)\sin^2(\theta)\sin(2\theta) \end{aligned} \quad (24)$$

Here one can see that the condition for the negative conditional isometric process $\mathbf{d} < \mathbf{d}_0$ in the above equation is

$$\sin^2\phi\sin 2\theta < 0 \quad (25)$$

This means that one has to choose the initial states to obtain the negative conditional isometric process by Gross-Pitaevskii nonlinearity. Thus, this model does not satisfy the initial state universality of the conditional isometric process. Initial state universality means that the process does not have requirement on special selection for the initial states, which is very important for applications. However, it should be praised that they succeeded in demonstrating the reduction of inner product for two nonorthogonal states by the above equation, and provided a positive answer to the author's prediction in the 1980s. Consequently the results of Childs-Young may provide a great progress in quantum communication theory, because two nonorthogonal states can be distinguished by such a real physical process. However, for applying this to optical quantum communications and others, the issues in the next section should be clarified.

V. Universality of nonlinear quantum mechanics

A. Formulation

Here I go back to the 1980s. At that time, I wanted to consider the quantum mechanics for which there is a non unitary evolution, for instance, Schrödinger equation with nonlinear terms. The possibility of reality of nonlinear quantum mechanics was pointed out by S.Weinberg [10,11] and others. Weinberg proposed the following equation from the standpoint testing the linearity of quantum mechanics, which is the general nonlinear Schrödinger equation keeping the homogeneity condition and without keeping the superposition condition [10].

$$i\frac{d\psi_k}{dt} = \frac{\partial h(\psi, \psi^*)}{\partial \psi_k^*} \quad (26)$$

In this system, if ψ_1 and ψ_2 are solutions, then $\lambda\psi_1$ is also solution for complex number λ, but $\psi_1 + \psi_2$ is not. This equation reduces to the ordinary linear Schrodinger equation when the real function $h(\psi, \psi^*)$ has a bilinear form $h = \psi_k^* H_{kl}\psi_l$. Some groups carried out the tests of the linearity of quantum mechanics experimentally by means of Ramsey's resonance methods or hydrogen maser. The results revealed to us a very small upper bound for the strength of the nonlinearity. On one hand, Kamesberger and Zeilinger investigated the effect of nonlinearity on Fresnel pattern by numerical simulation [14]. Thus, Morikaw-Takahara-Hirota [7] did not have sufficient knowledge on the nonlinearity in quantum mechanics in the 1980s except for the above. So we tried to employ nonlinear Schrödinger equation from a quantum field theoretic point of view to clarify quantum nonlinear effect in optical process as mentioned in the introduction.

In 2013, Meyer and Wong introduced the nonlinearity by Gross [15] and Pitaevskii[16] to consider quantum search problem [17]. This is an epoch-making for reality of nonlinear quantum mechanics, because Gross-Pitaevskii nonlinear equation is indeed initial state dependent dynamics which corresponds to conditional isometric process. Thus, it may be reasonable in view of conditional isometric process that Childs and Young reached their idea such as application of Gross-Pitaevskii nonlinear equation to discrimination issue of quantum states. Of course, this discovery is attributed to their achievement. The remaining problem is how much Gross-Pitaevskii model is universal. Basically, the nonlinear term (Eq-12) is related with Bose-Einstein condensates such as

$$\frac{h^2 a_s}{\pi m} N_0 |\psi(r,t)|^2 \qquad (27)$$

where m is the mass of the condensate atom, N_0 is the number of condensate atom, and a_s is boson-boson scattering length, respectively. At present nobody knows whether one can apply such a nonlinearity to optical process which is essential in optical quantum communication.

B. Time constant limitation

Let us here consider the practical quantum communication. The received quantum state control system in our sense is that received states are transformed by the conditional isometric process [6]. In communication theory, a time constant performance is the most important parameter. Information conveyed by quantum state is evaluated by bits per second. In general, the bit rate is more than 10^9 bits per second in the conventional communication. For the real time processing of the conditional isometric process, one requires the processing time less than 10^{-9} second.

In the conditional isometric process based on Gross-Pitaevskii model, the process time to distinguish the states is [13]

$$t_{\perp}(1) \cong \frac{1}{2g} \log \frac{1+\cos(\alpha_0/2)}{1-\cos(\alpha_0/2)} \qquad (28)$$

where g means the strength of the nonlinearity. In general nonlinearity model, one has [13]

$$t_{\perp}(2) \cong \frac{1}{g} \log \frac{1}{\epsilon} \qquad (29)$$

when $\epsilon << 1$ and $\cos\alpha_0 \cong 1-\epsilon$. So to satisfy the requirement $t_{\perp} < 10^{-9}$ in communication system is difficult. That is, when $\epsilon << 1$, one needs huge nonlinearity to provide $t_{\perp} < 10^{-9}$. This time constant property and the initial state universality property discussed in the section 4 may give a physical limitation of applications of nonlinear quantum mechanics to real communication problem. Thus, at present, one cannot confirm whether the nonlinear quantum mechanics gives serious impact on quantum communication theory or not.

VI. Concluding remark

In this paper, the importance of the results of Childs-Young theory in quantum sciences has been discussed. Their work is a milestone for a fundamental problem of quantum mechanics. Although the universality of nonlinear quantum mechanics is not yet clear at present, it is worth to investigating as a fundamental problem. The author expects that researchers will try to clarify the general theory on conditional isometric process based on nonlinear quantum mechanics including initial state dependent dynamics in the optical process.

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