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***Abstract*—The error probability of $2M$-level intensity modulated coherent state signal with base level offset is investigated under the assumptions that the signal is transmitted through a noisy channel and directly detected. For the $2M$-level signal, a simple approximation of the error probability by direct detection against the signal-to-noise ratio is derived by means of Gaussian approximation.**

I. Introduction

There are various types of signal modulation formats for current and future optical communication systems (*e.g.*, [1], [2], [3], [4]). In typical scenarios of optical communications, signal formats are appropriately chosen for realizing high-capacity, long-distant, and energy-efficient systems in accordance with system specifications and requirements. Particularly, the application range of multi-ary optical signals has become wider with the progress of digital coherent optical communication systems [5]. In addition, the design of signal modulation formats for secure optical communications is also important. As for the quantum stream cipher by Yuen 2000 protocol [6], [7] (abbreviated called as Y00; also known as $\alpha\eta$), its first experimental demonstration was done by using a multi-ary phase-shift keying signal [8] (See also [9]). Soon after the publication of Yuen's epoch-making work, Hirota proposed a multi-level intensity modulation-based implementation scheme of Y00 [10], [11], [12]. With this implementation scheme, various types of experiments of Y00 have been reported for investigating not only fundamental features of it ([12], [13], *etc*) but also practical communication performance as a secure optical communication system ([14], [15], [16], [17], [18], *etc*).

There are two remarkable discussions for further development of the quantum stream cipher; one is about a randomization technique called the quantum diffusion mapping (QDM) by Hirota and Kurosawa [19] (See also [20]), and another the coherent pulse position modulation (CPPM) (which was originally proposed in the literature [7] by Yuen and restated in the literature [21]) by Sohma and Hirota [22], [23], [24], [25]. Based on these discussions, a new concept of physical cipher called Quantum Enigma Cipher (QEC) was coined by Hirota [26]. Broad and specific reviews of QEC were made in the literatures [27](with Futami), [28], [29], [30] by himself. Further, in line with the progress of such theoretical research, the experimental research of the quantum stream cipher has entered a new stage with a newly developed transceiver named as *TU Cipher-0* [31], [32].

In the literature [20], we observe that the error probabilities (or the corresponding correct detection probabilities) of Y00 coherent state signal for an adversary who attempts to read true message or get a secret key were investigated by means of the approximation under several attacking scenarios. For example, the error probabilities of neighboring signals by heterodyne measurement were approximately evaluated for the phase, amplitude, and intensity modulation cases, respectively, and the approximations of the correct detection probabilities of Y00 coherent state signal by the square-root measurement at the single slot and code-word block cases were respectively derived under the condition that the absolute value of the inner product of neighboring signal states is close to unity. Since these analyses were done for proving the immunity against correlation and algebraic attacks, the resulting approximations illustrate the essential aspects of its cryptographic performance.

Among the signals used in the quantum stream cipher, we focus on the multi-level intensity modulated coherent state signal in this article. A notable feature of this signal is that it has a base level offset. As mentioned above, the cryptographic and communication performances of the intensity modulation-based Y00 as a secure optical communication system have been widely discussed. In contrast, the communication performance of this signal in a standard communication scenario where the signal conveys multi-bit information has not been discussed yet. Therefore, our question is how much communication performance can be expected when the multi-level signal with base level offset is directly utilized for the transmission of multi-bit information, not for cryptographic use: the purpose of our study is to investigate the error probability of the multi-level signal with base level offset in the standard communication scenario above.

For the analysis of the multi-level signal with base level offset to be done in this article, we suppose each signal state that forms the multi-level signal with base level offset is degraded by additional noise due to the environment, the optical amplifiers chaining the channel, and so on. In such cases, the quality of received signals is commonly measured by the signal-to-noise ratio. As the first step of our study on the multi-level signal with base level offset, the case of direct detection will be considered. That is, the error probability of the multi-level signal with base level offset by direct detection against the signal-to-noise ratio will be investigated. In addition, we will see that the exact form of the error probability of the multi-level signal is inconvenient for computation. Hence, we will attempt to find the approximation of the error probability by means of Gaussian approximation, and further, the resulting approximation will be compared with some exact values of the error probability to see the validity of the approximation.

II. A MODEL OF OUTPUT STATE FROM A NOISY CHANNEL

Suppose an optical signal in the coherent state $|\mu'\rangle$ of complex amplitude μ' is transmitted via a noisy communication channel, and the output state from the channel is expressed in the following form.

$$\hat{\rho}(\mu) = \frac{1}{\pi N}\int \exp[-\frac{|\gamma-\mu|^2}{N}]|\gamma\rangle\langle\gamma|\mathrm{d}^2\gamma, \quad (1)$$

where $|\gamma\rangle$ is the coherent state, and N is the average of the photon number of noise. Such a state actually appears in the discussions of the effect of thermal background noise and the analysis of phase insensitive amplifiers.

The photon distribution of this state is given [33] by

$$\begin{aligned} p(n|\mu) &= \langle n|\hat{\rho}(\mu)|n\rangle \\ &= \frac{N^n}{(1+N)^{n+1}} \\ &\quad \times \exp\left[-\frac{|\mu|^2}{1+N}\right] \\ &\quad \times \mathsf{L}_n\left[-\frac{|\mu|^2}{N(N+1)}\right], \end{aligned} \quad (2)$$

where $\mathsf{L}_n[x]$ is the Laguerre polynomial [34] (*Appendix A*). This distribution is commonly known as the Laguerre distribution. The average and variance of the photon number are respectively given [33] by

$$\langle\hat{n}\rangle = \mathrm{Tr}\ \hat{\rho}(\mu)\hat{a}^\dagger\hat{a} = |\mu|^2 + N, \quad (3)$$

and

$$\begin{aligned} \langle\Delta\hat{n}^2\rangle &= \langle\hat{n}^2\rangle - \langle\hat{n}\rangle^2 \\ &= \mathrm{Tr}\ \hat{\rho}(\mu)\hat{a}^\dagger\hat{a}\hat{a}^\dagger\hat{a} - \left(\mathrm{Tr}\ \hat{\rho}(\mu)\hat{a}^\dagger\hat{a}\right)^2 \\ &= |\mu|^2 + 2|\mu|^2N + N(N+1). \end{aligned} \quad (4)$$

The signal-to-noise ratio of direct detection for the state $\hat{\rho}(\mu)$ is given by

$$\mathsf{SNR} = \frac{\langle\hat{n}\rangle^2}{\langle\Delta\hat{n}^2\rangle} = \frac{(|\mu|^2+N)^2}{|\mu|^2+2|\mu|^2N+N(N+1)}. \quad (5)$$

III. $2M$-LEVEL SIGNAL WITH BASE LEVEL OFFSET

Here we give a mathematical description of $2M$-level signal with base level offset in the presence of additional noise. Suppose the signal intensity levels, $S_0, S_1, \ldots S_{2M-1}$, are placed ΔS photons apart, the base level S_0 is nonzero, and each quantum state of the received signals in front of a detector is given as $\hat{\rho}(\sqrt{S_i})$. Defining the parameters

$$\bar{S} = \frac{1}{2M}\sum_{i=0}^{2M-1} S_i \quad (6)$$

and

$$r = \frac{S_{2M-1}}{S_0}, \quad (7)$$

the signal intensity levels are given by

$$S_i = \left\{\frac{2}{r+1} + i\frac{2(r-1)}{(r+1)(2M-1)}\right\}\bar{S}. \quad (8)$$

A schematic of the signal intensity level structure defined above is illustrated in Fig. 1, where the intensity levels within the interval from S_0 to S_{2M-1} have been omitted.

The average intensity level of a received signal is given by

$$S_i' = S_i + N, \quad (9)$$

and the corresponding variance is given by

$$\sigma_i^2 = S_i + 2S_iN + N(N+1). \quad (10)$$

When the thresholds for decision are the middle levels between the closest received signals, the probabilities of correct decision are given as follows.

$$P(i|i) = \sum_{n=\ell(i)}^{u(i)} p(n|\sqrt{S_i}), \quad (11)$$

where

$$u[i] = \begin{cases} \lceil (S_i' + S_{i+1}')/2 \rceil, & i < 2M-1, \\ \infty, & i = 2M-1, \end{cases} \quad (12)$$

and

$$\ell[i] = \begin{cases} 0, & i = 0, \\ u[i-1]+1, & i > 0. \end{cases} \quad (13)$$

Assuming that the element signals are equiprobable, the average probability of error is given by

$$\bar{P}_{\mathrm{e}} = 1 - \frac{1}{2M}\sum_{i=0}^{2M-1} P(i|i). \quad (14)$$

To characterize the multi-level signal by one parameter, the averaged state over all the element signals is introduced as follows.

$$\hat{\rho} = \frac{1}{2M}\sum_{i=0}^{2M-1} \hat{\rho}(\sqrt{S_i}). \quad (15)$$

The average and variance of the photon number for the state $\hat{\rho}$ are respectively given as follows.

$$\langle \hat{n} \rangle|_{\hat{\rho}} = \mathrm{Tr}\ \hat{\rho}\hat{a}^{\dagger}\hat{a} = \bar{S} + N \tag{16}$$

and

$$\langle \Delta\hat{n}^2 \rangle\big|_{\hat{\rho}} = \bar{S} + 2\bar{S}N + N(N+1). \tag{17}$$

From these quantities, the signal-to-noise ratio for the state $\hat{\rho}$ is given by

$$\mathsf{SNR}|_{\hat{\rho}} = \frac{(\bar{S}+N)^2}{\bar{S}+2\bar{S}N+N(N+1)}. \tag{18}$$

IV. Approximation of the error probability

A. Gaussian approximation

The exact value of the error probability for a multi-level signal is obtained from Eqs.(11)-(14) and Eq.(18). However, this formula is inconvenient for computation because a number of $p(n|S_i)$ must be evaluated when $\bar{S}$ is large. This motivates us to find an approximation of Eq.(14) for large $\bar{S}$.

To find the approximation, we first assume that each $p(n|S_i)$ can be approximated as the normal distribution with mean S_i' and variance σ_i^2. Then the probabilities of correct decision of Eq.(11) are approximately expressed as follows.

$$P(i|i) \approx \begin{cases} 1-\mathsf{Q}[\Delta S/2\sigma_0], & i=0; \\ 1-2\mathsf{Q}[\Delta S/2\sigma_i], & 0<i<2M-1; \\ 1-\mathsf{Q}[\Delta S/2\sigma_{2M-1}], & i=2M-1, \end{cases} \tag{19}$$

where $\mathsf{Q}[x]$ is the Q-function (*Appendix C*). Therefore, the error probability is rewritten as

$$\begin{aligned} \bar{P}_{\mathrm{e}} \approx\ & \frac{1}{2M}\mathsf{Q}[\Delta S/2\sigma_0] \\ & +\frac{1}{M}\sum_{i=1}^{2M-2}\mathsf{Q}[\Delta S/2\sigma_i] \\ & +\frac{1}{2M}\mathsf{Q}[\Delta S/2\sigma_{2M-1}]. \end{aligned} \tag{20}$$

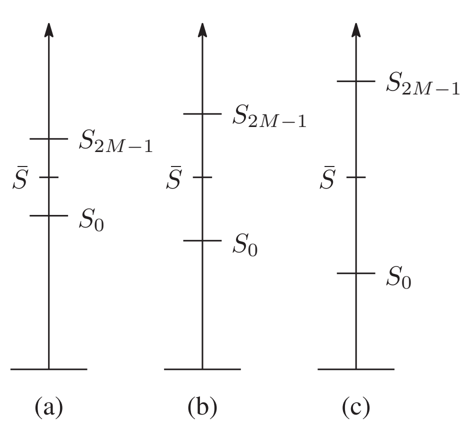

Fig. 1. (a) $r = 1.5$, (b) $r = 2$, (c) $r = 3$

Each variance σ_i^2 depends on the signal intensity level S_i, and S_i is obtained by the average intensity level $\bar{S}$ and th index i as shown in Eq.(8). Hence, when $\bar{S} \gg N$, we observe

$$\frac{\Delta S}{2\sigma_i} \approx K_i\sqrt{\mathsf{SNR}|_{\hat{\rho}}}, \tag{21}$$

where

$$K_i = \frac{r-1}{\sqrt{2(2M-1)(r+1)\{(2M-1)+i(r-1)\}}}. \tag{22}$$

Substituting this into Eq.(20), we obtain

$$\begin{aligned} \bar{P}_{\mathrm{e,approx}} =\ & \frac{1}{2M}\mathsf{Q}[K_0\sqrt{\mathsf{SNR}|_{\hat{\rho}}}] \\ & +\frac{1}{M}\sum_{i=1}^{2M-2}\mathsf{Q}[K_i\sqrt{\mathsf{SNR}|_{\hat{\rho}}}] \\ & +\frac{1}{2M}\mathsf{Q}[K_{2M-1}\sqrt{\mathsf{SNR}|_{\hat{\rho}}}]. \end{aligned} \tag{23}$$

B. Validity of the approximation

Numerical evaluation of $\bar{P}_{\mathrm{e}}$ and $\bar{P}_{\mathrm{e,approx}}$ is done in the following conditions:

- M: 2 and 4 (4-level and 8-level)
- $\bar{S}$: 10 000 photons
- r : 1.5, 2.0, and 3.0
- $\mathsf{SNR}_{\mathrm{dB}}$: 20 $\sim$ 35 dB for $\bar{P}_{\mathrm{e}}$, and 20 $\sim$ 40 dB for $\bar{P}_{\mathrm{e,approx}}$

The results of numerical simulations for 4-level and 8-level signals at $r = 1.5, 2, 3$ are shown in Fig.2 and Fig.3, respectively. In each case, the dots stand for the exact error probability $\bar{P}_{\mathrm{e}}$, and the solid lines are $\bar{P}_{\mathrm{e,approx}}$. In addition, the difference and relative difference between $\bar{P}_{\mathrm{e}}$ and $\bar{P}_{\mathrm{e,approx}}$ are shown in Fig. 4. Recall that the error probability is usually plotted in log or double-log scale. Taking account of this fact, the difference between $\bar{P}_{\mathrm{e}}$ and $\bar{P}_{\mathrm{e,approx}}$ is small enough. Thus, we see that $\bar{P}_{\mathrm{e,approx}}$ gives a good approximation to $\bar{P}_{\mathrm{e}}$.

C. An application of the approximation

For the demonstration of the usefulness of the approximation, let us use the formula (23) to estimate the signal-to-noise ratio SNR required to achieve the designed error probability $\bar{P}_{\mathrm{e}}$ at a given parameter r. To do so, we first set $\bar{P}_{\mathrm{e}} = 10^{-9}$ for 4-level signal ($M = 2$), and $\bar{P}_{\mathrm{e}} = 10^{-5}$ for 8-level signal ($M = 4$). By using the formula (23) the relations between the signal-to-noise ratio and the parameter r for these two cases are easily obtained as shown in Fig.5 and Fig.6, respectively.

V. Summary

The error probability of $2M$-level intensity modulated coherent state signal with base level offset was investigated under the assumptions that the signal is transmitted through a noisy channel and directly detected. Based on a model of the received element signals of the $2M$-level signal in the presence of additional noise, the error probability for the $2M$-level signal by direct detection was formally obtained. However, the exact formula of the error probability we obtained is inconvenient for computation due to its computational load when the average signal level is large, so that a simple approximation of the error probability was derived by means of Gaussian approximation. To see the validity of the approximation, it was numerically compared with some exact values of the error probability in the case of 4-level and 8-level signals. From this simulation, it was shown that the difference between the exact value and the approximation is small enough.

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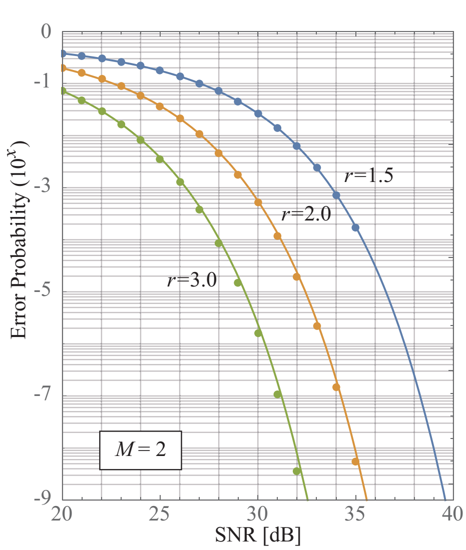

Fig. 2. 4-level signal ($M = 2$)

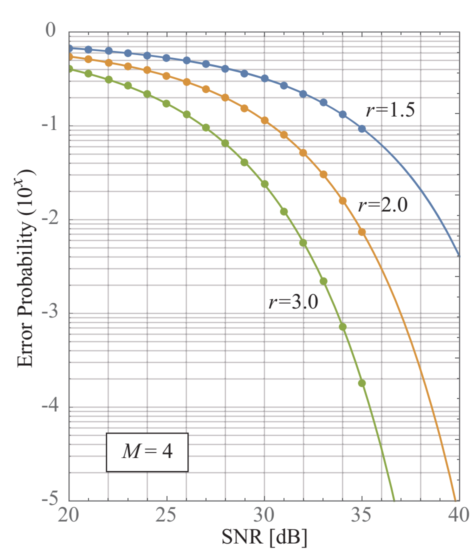

Fig. 3. 8-level signal ($M = 4$)

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APPENDIX

A. The Laguerre polynomials ([34])

$$\mathsf{L}_n[x] = \sum_{k=0}^{n} \binom{n}{k} \frac{(-1)^k}{k!} x^k.$$

For instance, $\mathsf{L}_0[x] = 1$, $\mathsf{L}_1[x] = -x + 1$, and $\mathsf{L}_2[x] = \frac{1}{2}(x^2 - 4x + 2)$.

B. The complementary error function ([34])

$$\mathsf{erfc}[x] = \frac{2}{\sqrt{\pi}} \int_x^\infty \mathrm{e}^{-\tau^2} \mathrm{d}\tau.$$

C. The Q-function ([35])

$$\mathsf{Q}[x] = \frac{1}{\sqrt{2\pi}} \int_x^\infty \exp[-\frac{\tau^2}{2}] \mathrm{d}\tau = \frac{1}{2} \mathsf{erfc}[\frac{x}{\sqrt{2}}].$$

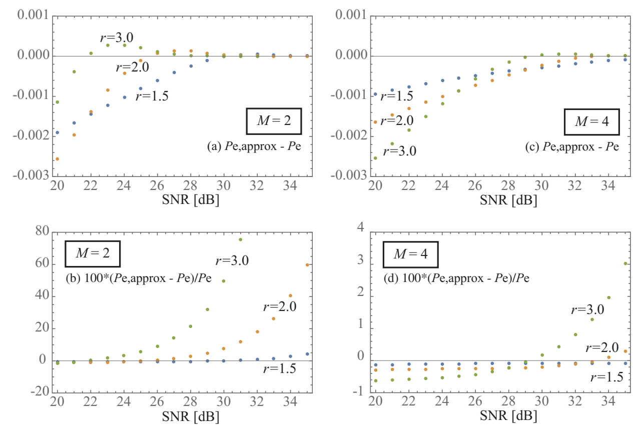

Fig. 4. (a) The difference $\bar{P}_{\mathrm{e,approx}} - \bar{P}_{\mathrm{e}}$ and (b) the relative difference $100 \times (\bar{P}_{\mathrm{e,approx}} - \bar{P}_{\mathrm{e}})/\bar{P}_{\mathrm{e}}$ for 4-level signal ($M = 2$); (c) The difference $\bar{P}_{\mathrm{e,approx}} - \bar{P}_{\mathrm{e}}$ and (d) the relative difference $100 \times (\bar{P}_{\mathrm{e,approx}} - \bar{P}_{\mathrm{e}})/\bar{P}_{\mathrm{e}}$ for 8-level signal ($M = 4$)

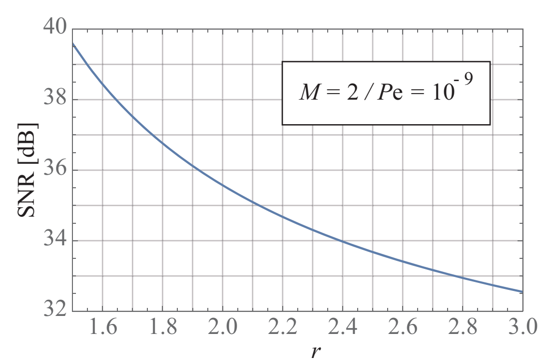

Fig. 5. SNR versus r for 4-level signal ($M = 2$) at $\bar{P}_{\mathrm{e}} = 10^{-9}$

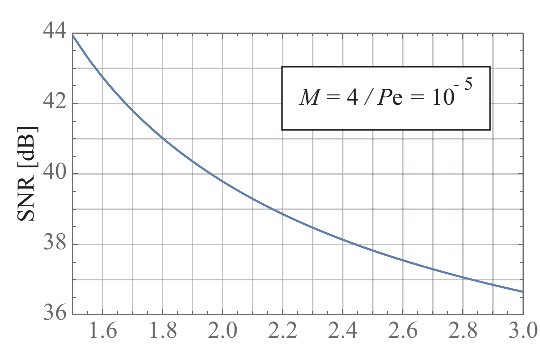

Fig. 6. SNR versus r for 8-level signal ($M = 4$) at $\bar{P}_{\mathrm{e}} = 10^{-5}$