ISSN 2186-6570

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Tamagawa University Quantum ICT Research Institute Bulletin, Vol.1, No.1, 33-40, 2011

Tamagawa University Quantum ICT Research Institute Bulletin
Vol.1 No.1 : 33-40 (2011)

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***Abstract*—The optimal distributions on a channel input that yield the random coding exponent, the expurgation exponent, and the zero-rate reliability function for a discrete classical-quantum channel with M-ary Phase Shift Keying (PSK) coherent state signal are analytically derived. In each case, the optimal distributions are given by a uniform distribution over the channel input alphabet. As a result, the expressions of the error exponents for M-ary PSK coherent state signal are simplified. By using this result, typical behavior of the error exponents is sketched in the case of $M = 16$.**

I. Introduction

It is well-known that an elementary proof of the channel coding theorem in conventional information theory is given by the lower bounds of the reliability function [1]. Like in the conventional case, the reliability function based proof of the channel coding theorem for a discrete classical-quantum channel with pure states was established by Burnashev and Holevo [2]. In their paper, the random coding exponent and the expurgation exponent for a discrete classical-quantum channel with pure states have been formulated as the lower bounds of the associated reliability function. Such error exponents provide not only the simple proof of the channel coding theorem but also useful tools for evaluating communication systems. However, the error exponents which have been respectively formulated in Refs [1] and [2] involves an optimization problem of a certain function with respect to *a priori* distributions on the input alphabet and with respect to another single parameter, so that computation of the error exponents would be complicated in general.

Fundamental features of M-ary Phase Shift Keying (PSK) coherent state signal have been widely discussed; For example, the minimum error detection process for a "homogeneous" set of pure states was analytically derived by Belavkin [3]. Independently, Ban *et al.* clarified that the minimum error detection process for "symmetric" pure states is the square-root measurement. In both cases, M-ary PSK coherent state signal becomes a good example for their practical applications. Further, the error performance of the square-root measurement for M-ary PSK coherent state signal was investigated, together with the case of quadrature amplitude modulation (QAM) signal in Ref.[5]. The closed-form expression of the channel capacity for a discrete classical-quantum channel with M-ary PSK coherent state signal was given in Ref.[6]. Thus the communication system with M-ary PSK coherent state signal is an important model in quantum communication theory. However, the evaluation of quantum communication systems with M-ary PSK coherent state signal from the point of view of the reliability function is still remaining except for some cases [7], [8]. To apply the theory of the reliability function for evaluating the quantum communication systems with M-ary coherent state signal, computation of the error exponents is of fundamental interest. Therefore the purpose of this paper is to give simplified expressions of the error exponents for M-ary coherent state signal in order to reduce computation procedures. The main task is to find the optimal distributions yielding the error exponents — the random decoding exponent, the expurgation exponent, and the zero-rate reliability function — for a discrete classical-quantum channel with M-ary PSK coherent state signal.

This paper is organized as follows. In section II, we summarize the general theory of the reliability function for a discrete classical-quantum channel with pure state signal, which was formulated by Burnashev and Holevo. In section III, we investigate quantum communication system with M-ary PSK coherent state signal. It is shown that the optimal distributions of the error exponents for a discrete classical-quantum channel with M-ary PSK coherent state signal are respectively given by a uniform distribution on the input alphabet. As a result, the expressions of the error exponents for a discrete classical-quantum channel with M-ary PSK coherent state signal are simplified. To illustrate the typical behavior of the error exponents we present a numerical example in section IV. In section V, we summarize our results.

II. Reliability Function for a Discrete Classical-Quantum Channel with Pure State Signal

Let $\mathcal{A} = \{1, 2, \ldots, M\}$ be an input alphabet of a classical-quantum channel and $\mathcal{B} = \{|\psi_1\rangle, |\psi_2\rangle, \cdots, |\psi_M\rangle\}$ a set of state vectors in Hilbert space $\mathcal{H}$. Then, a discrete classical-quantum channel with pure state signal is characterized by a mapping $k \in \mathcal{A} \mapsto |\psi_k\rangle \in \mathcal{B}$. Since we use the coherent states

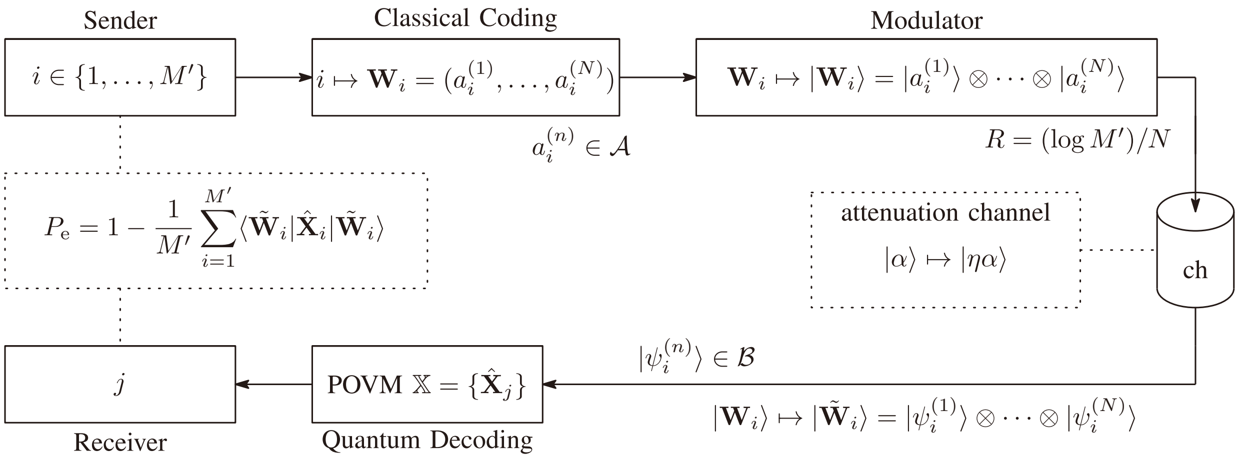

Fig. 1. Communication model

of light for signaling, this channel model is applicable to practical situation because the coherent state is the only state that keeps its purity of quantum state after passing a communication channel with attenuation [9] (Fig. 1).

An input for this discrete classical-quantum channel is described by *a priori* probability distribution $\mathbf{p} = (p_1, p_2, \ldots, p_M)$ on the input alphabet $\mathcal{A}$. According to the channel coding theorem for a discrete classical-quantum channel with pure state signal [2], [10] (See also the general description [11], [12], [13], [14]), the channel capacity for this classical-quantum channel is given by

$$C = \max_{\mathbf{p}}\left[H\Big(\sum_{k=1}^{M} p_k|\psi_k\rangle\langle\psi_k|\Big)\right], \tag{1}$$

where $H(\hat{\rho}) \equiv -\mathrm{Tr}\hat{\rho}\ln\hat{\rho}$ is the von Neumann entropy for a density operator $\hat{\rho}$. This quantity means the quantum limit of error-free transmission rate for the classical-quantum channel above. That is, if the transmission rate is less than the channel capacity C, then there exists a zero-error code for this channel. Alternatively, the channel coding theorem for a discrete classical-quantum channel with pure state signal can be represented with the so-called reliability function. The reliability function is defined by

$$E(R) \equiv \limsup_{N\to\infty}\left[\frac{-\ln P_{\rm e}(N,R)}{N}\right] \tag{2}$$

for $0 < R < C$, where N is the length of a codeword, R is the transmission rate, and $P_{\rm e}(N,R)$ is the minimum probability of decoding error. For a discrete classical-quantum channel with pure state signal, the minimum probability of decoding error is written as

$$P_{\rm e}(N,R) = \inf_{\mathbb{W},\mathbb{X}}\left[\frac{1}{M'}\sum_{i=1}^{M'}(1-\langle\tilde{\mathbf{W}}_i|\hat{\mathbf{X}}_i|\tilde{\mathbf{W}}_i\rangle)\right], \tag{3}$$

where $\mathbb{W}$ is a codebook defined by

$$\mathbb{W} = \Big\{|\tilde{\mathbf{W}}_i\rangle = |\psi_i^{(1)}\rangle \otimes |\psi_i^{(2)}\rangle \otimes \cdots \otimes |\psi_i^{(N)}\rangle : i = 1,2,\ldots,M',\ |\psi_i^{(n)}\rangle \in \mathcal{B}\Big\}, \tag{4}$$

and M' is the size of the codebook, and where the decoding process $\mathbb{X}$ is represented by a positive operator-valued measure (POVM)

$$\mathbb{X} = \Big\{\hat{\mathbf{X}}_j :\ \hat{\mathbf{X}}_j \geq 0\ \forall j,\ \sum_{\text{all } j}\hat{\mathbf{X}}_j = \hat{1}^{(N)}\Big\}, \tag{5}$$

and $\hat{1}^{(N)}$ is the identity operator on the N-th tensor of the signal Hilbert space $\mathcal{H}^{\otimes N}$. Although it is difficult to derive the exact form of the reliability function $E(R)$ for all rate, the zero-rate reliability function has been given as follows [2]: if $|\langle\psi_k|\psi_l\rangle| > 0$ for any pair (k,l), then the zero-rate reliability function becomes

$$E(+0) = -\min_{\mathbf{p}}\sum_{k=1}^{M}\sum_{l=1}^{M} p_k p_l \ln|\langle\psi_k|\psi_l\rangle|^2, \tag{6}$$

and if $|\langle\psi_k|\psi_l\rangle| = 0$ for some pair (k,l), then

$$E(+0) = +\infty. \tag{7}$$

Thus, the exact form of the reliability function $E(R)$ for a discrete classical-quantum channel with pure state signal is still open except for the case of $R = +0$. However, the lower bounds of the reliability function of this channel have been well formulated for all rate. From this point, we mention the basic results on the lower bounds [2]. By using the random coding technique, the relation

$$P_{\rm e}(N,R) \leq 2\exp[-N(E_0(\mathbf{p};s) - sR)] \tag{8}$$

holds for $0 \leq s \leq 1$, where

$$E_0(\mathbf{p};s) \equiv -\ln\mathrm{Tr}\left[\sum_{k=1}^{M} p_k|\psi_k\rangle\langle\psi_k|\right]^{1+s}. \tag{9}$$

Then the random coding exponent of a discrete classical-quantum channel with pure state signal is defined by

$$E_r(R) \equiv \max_{0\le s\le 1} \max_{\mathbf{p}} \Big[E_0(\mathbf{p};s) - sR\Big] \tag{10}$$

for $0 < R < C$. This function satisfies the relation $E(R) \geq E_r(R)$. Furthermore, by expurgating poor codewords from the codes in an ensemble, we have

$$P_e(N,R) \le \exp\Big[sN(R+\frac{\ln 4}{N})\Big] \times \left(\sum_{k=1}^{M}\sum_{l=1}^{M} p_k p_l |\langle\psi_k|\psi_l\rangle|^{2/s}\right)^{sN} \tag{11}$$

for $s \geq 1$. Then the expurgation exponent for a discrete classical-quantum channel with pure state signal is defined by

$$E_{ex}(R) \equiv \max_{s\ge 1} \max_{\mathbf{p}} \Big[E_x(\mathbf{p};s) - sR\Big] \tag{12}$$

for $0 < R \leq R_c$, and

$$E_{ex}(R) \equiv 0 \tag{13}$$

for $R_c \leq R < C$, where

$$E_x(\mathbf{p},s) \equiv -s \ln \sum_{k=1}^{M}\sum_{l=1}^{M} p_k p_l |\langle\psi_k|\psi_l\rangle|^{2/s}, \tag{14}$$

and the cutoff rate

$$R_c \equiv \max_{\mathbf{p}} E_x(\mathbf{p},1). \tag{15}$$

In this case, it satisfies $E(R) \geq E_{ex}(R + (\ln 4)/N)$.

III. Optimal Distribution of the Error Exponents for M-ary PSK Coherent State Signal

In this section we consider the case of an M-ary PSK coherent state signal. The signal set is given by

$$\mathcal{B} = \Big\{|\psi_k\rangle = |\alpha \exp[\mathbf{i}\frac{2\pi(k-1)}{M}]\rangle \; : k = 1,2,\ldots,M\Big\}, \tag{16}$$

where $|\alpha\rangle$ stands for a coherent state of light having complex amplitude α. (See Fig. 2). Then each signal can be rewritten as

$$|\psi_k\rangle = \hat{V}^{k-1}|\alpha\rangle, \quad \forall k, \tag{17}$$

where the unitary operator $\hat{V} \equiv \exp[\mathbf{i}(2\pi/M)\hat{a}^\dagger\hat{a}]$, $\mathbf{i} \equiv \sqrt{-1}$, and $\hat{a}, \hat{a}^\dagger$ are the bosonic annihilation and creation operators, respectively. The average number of photons per signal for M-ary PSK is given by $|\alpha|^2$, which is depending not on the *a priori* distribution. The inner product between two signals is given by

$$\langle\psi_k|\psi_l\rangle = A_{(k,l)} \exp[\mathbf{i}\Theta_{(k,l)}], \quad \forall(k,l), \tag{18}$$

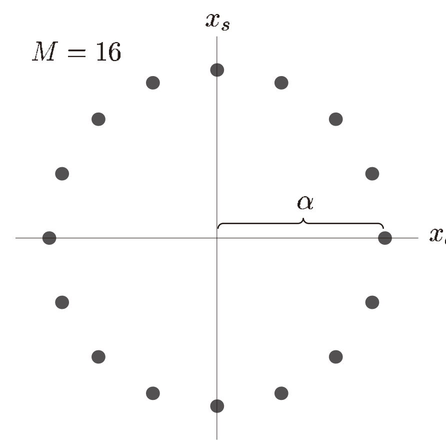

Fig. 2. Signal constellation of 16-PSK in (x_c, x_s)-plane ($\alpha > 0$)

with

$$A_{(k,l)} = \exp\left[-2|\alpha|^2 \sin^2\Big[\frac{\pi}{M}(l-k)\Big]\right], \tag{19}$$

$$\Theta_{(k,l)} = |\alpha|^2 \sin\Big[\frac{2\pi}{M}(l-k)\Big]. \tag{20}$$

Here we derive the optimal distribution yielding the random coding exponent for the M-ary PSK coherent state signal. First we let

$$\hat{\rho}(\mathbf{p}) \equiv \sum_{k=1}^{M} p_k |\psi_k\rangle\langle\psi_k|. \tag{21}$$

Consider an arbitrary *a priori* distribution $\mathbf{p} = (p_1, p_2, \ldots, p_M) \equiv \mathbf{p}^{(1)}$ and its permutations

$$\begin{cases} \mathbf{p}^{(2)} &= (p_2, p_3, \ldots, p_1), \\ \mathbf{p}^{(3)} &= (p_3, p_4, \ldots, p_2), \\ &\vdots \\ \mathbf{p}^{(M)} &= (p_M, p_1, \ldots, p_{M-1}). \end{cases} \tag{22}$$

Then we have $\hat{\rho}(\mathbf{p}^{(k)}) = \hat{V}^{k-1}\hat{\rho}(\mathbf{p})\hat{V}^{\dagger k-1}$, $k = 1,2,\ldots,M$. From this, $E_0(\mathbf{p}^{(k)};s) = E_0(\mathbf{p};s)$, $k = 1,2,\ldots,M$. Since $E_0(\mathbf{p};s)$ is a concave function of $\mathbf{p}$ for fixed s, we obtain

$$\begin{aligned} E_0(\mathbf{p};s) &= \frac{1}{M}\sum_{k=1}^{M} E_0(\mathbf{p}^{(k)};s) \\ &\le E_0(\sum_{k=1}^{M}\frac{1}{M}\mathbf{p}^{(k)};s) \\ &= E_0(\frac{1}{M},\frac{1}{M},\ldots,\frac{1}{M};s). \end{aligned} \tag{23}$$

Observe that this inequality holds for any s. Therefore the optimal distribution of the random coding exponent is given by a uniform distribution on the input. Now let us consider the maximization with respect to the parameter s. As stated in Ref.[2], the function $E_0(\mathbf{p};s)$

is nondecreasing and concave function of s in general. Moreover, by defining

$$E_r(\mathbf{p};R) \equiv \max_{0\le s\le 1}[E_0(\mathbf{p};s) - sR], \tag{24}$$

it becomes

$$E_r(\mathbf{p};R) = E_0(\mathbf{p};1) - R \tag{25}$$

for $0 \le R \le R_{cr}(\mathbf{p})$, where the critical rate $R_{cr}(\mathbf{p})$ is defined as

$$R_{cr}(\mathbf{p}) \equiv \left.\frac{\partial E_0(\mathbf{p};s)}{\partial s}\right|_{s=1}. \tag{26}$$

Therefore, the exact expression of the random coding exponent for the region $0 \le R \le R_{cr}(\mathbf{p})$ is given by

$$E_r(R) = \left(-\ln\sum_{k=1}^{M}\lambda_k^2\right) - R, \tag{27}$$

where λ_k, $k = 1,2,\ldots,M$, are the eigenvalues of the density operator $\hat{\rho}(1/M,1/M,\ldots,1/M)$ and the corresponding critical rate is given by

$$R_{cr} = -\frac{\sum_{k=1}^{M}\lambda_k^2\ln\lambda_k}{\sum_{k=1}^{M}\lambda_k^2}. \tag{28}$$

Hence we have the next proposition.

Proposition 1: The optimal distribution of the random coding exponent for M-ary PSK coherent state signal is given by a uniform distribution over the input alphabet. Then the random coding exponent for M-ary PSK coherent state signal is given by Eq.(27) for $0 < R < R_{cr}$, and

$$E_r(R) = \max_{0\le s\le 1}\left[\left(-\ln\sum_{k=1}^{M}\lambda_k^{1+s}\right) - sR\right], \tag{29}$$

for $R_{cr} \le R < C$, where the critical rate R_{cr} has been given in Eq.(28) and the channel capacity is given by

$$C = -\sum_{k=1}^{M}\lambda_k\ln\lambda_k, \tag{30}$$

and where the eigenvalues λ_k are given by

$$\lambda_k = \frac{1}{M}\sum_{l=1}^{M}A_{(1,l)}\cos\Big[\Theta_{(1,l)} - \frac{2\pi}{M}k(l-1)\Big], \tag{31}$$

where $A_{(1,l)}$ and $\Theta_{(1,l)}$ have been given in Eqs.(19) and (20), respectively. □

Second, we consider the optimal distribution yielding the expurgation exponent. Since the function $E_x(\mathbf{p};s)$ is not a concave function of $\mathbf{p}$ for fixed s (APPENDIX A), we cannot apply the optimization technique used in the above. So, we define a new function $F_1(\mathbf{p})$ to find the optimal distribution of the function $E_x(\mathbf{p},s)$ in Eq.(12) as follows:

$$\begin{aligned} F_1(\mathbf{p}) &\equiv \exp[-E_x(\mathbf{p};s)/s] \\ &= \sum_{k=1}^{M}\sum_{l=1}^{M}p_k p_l \exp\Big[-\frac{4|\alpha|^2}{s}\tilde{z}_{(k,l)}\Big], \end{aligned} \tag{32}$$

where we have set

$$\tilde{z}_{(k,l)} \equiv \sin^2\Big[\frac{\pi}{M}(l-k)\Big]. \tag{33}$$

If the optimal distribution that minimizes $F_1(\mathbf{p})$ is independent from the parameter s, then the maximization of the function $E_x(\mathbf{p};s)$ with respect to *a priori* distributions will be replaced to the minimization of the function $F_1(\mathbf{p})$. To see this, we examine the convexity of the function with respect to *a priori* distributions by using some useful results stated by Jelinek.

Definition (APPENDIX of Ref.[15]): Let $\mathbf{c} = (c_1, c_2, \cdots, c_M)$ be a zero-sum complex number pair such that $\sum_{k=1}^{M}c_k = 0$. An $M\times M$ matrix $\tilde{X} = [\tilde{x}_{(k,l)}]$ is said to be *negative almost-definite* if $\sum_{k=1}^{M}\sum_{l=1}^{M}c_k^*\tilde{x}_{(k,l)}c_l \le 0$ for any zero-sum complex number pair $\mathbf{c}$, where $*$ stands for the complex conjugate of a complex number.

Lemma (*Lemma 1a* of Ref.[15]): An $M\times M$ matrix $\tilde{X} = [\tilde{x}_{(k,l)}]$ is negative almost-definite if and only if the matrix $X = [x_{(k,l)}]$ with entries

$$\begin{aligned} x_{(k,l)} \equiv& -\tilde{x}_{(k,l)} + \frac{1}{M}\sum_{i=1}^{M}\tilde{x}_{(i,l)} \\ &+ \frac{1}{M}\sum_{j=1}^{M}\tilde{x}_{(k,j)} - \frac{1}{M^2}\sum_{i=1}^{M}\sum_{j=1}^{M}\tilde{x}_{(i,j)} \end{aligned} \tag{34}$$

is non-negative definite.

Theorem (*Theorem 2a* of Ref.[15]): For any positive number $\delta > 0$, a hermitian matrix $X' = [\exp[-\delta\tilde{x}_{(k,l)}]]$ is non-negative definite if and only if the matrix $\tilde{X} = [\tilde{x}_{(k,l)}]$ is negative almost-definite.

Using these results, the convexity of $F_1(\mathbf{p})$ is shown as follows: First we define

$$\begin{aligned} \Delta_{F_1} &\equiv tF_1(\mathbf{p}) + (1-t)F_1(\mathbf{p}') - F_1(t\mathbf{p} + (1-t)\mathbf{p}') \\ &= (1-t)t\sum_{k=1}^{M}\sum_{l=1}^{M}q_k q_l \exp\left[-\frac{4|\alpha|^2}{s}\tilde{z}_{(k,l)}\right], \end{aligned} \tag{35}$$

where $0 \le t \le 1$ and we have defined $\mathbf{q} \equiv \mathbf{p} - \mathbf{p}'$. Let us examine whether $\Delta_{F_1} \ge 0$ or not. From Eqs.(33) and (34), we compute a new matrix $Z \equiv [z_{(k,l)}]$ with

$$\begin{aligned} z_{(k,l)} \equiv& -\tilde{z}_{(k,l)} + \frac{1}{M}\sum_{i=1}^{M}\tilde{z}_{(i,l)} \\ &+ \frac{1}{M}\sum_{j=1}^{M}\tilde{z}_{(k,j)} - \frac{1}{M^2}\sum_{i=1}^{M}\sum_{j=1}^{M}\tilde{z}_{(i,j)} \\ =& \frac{1}{2}\cos[\frac{2\pi}{M}(l-k)], \end{aligned} \tag{36}$$

where we have used $\sum_{i=1}^{M}\tilde{z}_{(i,l)} = \sum_{j=1}^{M}\tilde{z}_{(k,j)} = M/2$. Since the matrix Z is a circular matrix, the eigenvalues of Z are given as

$$\eta_k = \begin{cases} M/4, & \text{if } k = 1, \\ M/4, & \text{if } k = M-1, \\ 0, & \text{elsewhere.} \end{cases} \tag{37}$$

Thus the matrix $Z = [z_{(k,l)}]$ is non-negative definite. From *Lemma 1a*, $Z \geq 0$ implies that the matrix $\tilde{Z} = [\tilde{z}_{(k,l)}]$ is negative almost-definite. Furthermore, this implies that the matrix $Z' = [\exp[-(4|\alpha|^2/s)\tilde{z}_{(k,l)}]]$ is non-negative definite. Therefore we have $\Delta_{F_1} \geq 0$, that is, the function $F_1(\mathbf{p})$ is a convex function of $\mathbf{p}$.

From the convexity of the function $F_1(\mathbf{p})$, we can easily find the optimal distribution of the function $F_1(\mathbf{p})$ with the same manner as the case of the random coding exponent. Actually, it is given by a uniform distribution $p_k = 1/M$ for any s. So, the remaining task is the maximization with respect to the parameter s. In general, it is difficult to obtain the optimal s analytically over all the rates below the cutoff rate. However, it has been proved in Ref.[2] that $E_{ex}(R) = E_r(R)$ if the rate R is lying in the range from the rate R'_{cr} given by

$$R'_{cr} = -\ln \sum_{k=1}^{M} (\lambda_k(\mathbf{p}^\circ))^2 + \frac{\sum_{k=1}^{M}\sum_{l=1}^{M} p_k^\circ p_l^\circ |\langle\psi_k|\psi_l\rangle|^2 \ln|\langle\psi_k|\psi_k\rangle|^2}{\sum_{k=1}^{M}(\lambda_k(\mathbf{p}^\circ))^2} \tag{38}$$

to the rate R_{cr} given in Eq.(28), where $\mathbf{p}^\circ$ is the optimal distribution of the expurgation exponent at the rate R'_{cr}, and $\lambda_k(\mathbf{p}^\circ)$ is the eigenvalue of the density operator $\hat{\rho}(\mathbf{p}^\circ)$. Moreover, it has been also proved [2] that, if the rate R is lying in the range form the rate R'_{cr} to the cutoff rate R_c, the expurgation exponent is given by $E_{ex}(R) = \max_{\mathbf{p}}[E_x(\mathbf{p};1) - R]$. In our case, since the optimal distribution is given by a uniform distribution over all the rates, the rate R'_{cr} becomes

$$R'_{cr} = -\ln \sum_{k=1}^{M} \lambda_k^2 - \frac{4|\alpha|^2}{M} \cdot \frac{\sum_{l=1}^{M} \tilde{z}_{(1,l)} \exp[-4|\alpha|^2 \tilde{z}_{(1,l)}]}{\sum_{k=1}^{M} \lambda_k^2} \tag{39}$$

and we can conclude that $E_{ex}(R)$ is a linear function of R in the range $[R'_{cr}, R_c)$. Hence we have the next proposition.

Proposition 2: The optimal distribution of the expurgation exponent for M-ary PSK coherent state signal is given by a uniform distribution. Then the expurgation exponent is given by

$$E_{ex}(R) = \max_{s\geq 1}\left[-sR - s\ln\frac{1}{M}\sum_{l=1}^{M}\exp\left[-\frac{4|\alpha|^2}{s}\tilde{z}_{(1,l)}\right]\right] \tag{40}$$

for $0 < R \leq R'_{cr}$, and

$$E_{ex}(R) = -\ln\frac{1}{M}\sum_{l=1}^{M}\exp\left[-4|\alpha|^2\tilde{z}_{(1,l)}\right] - R \tag{41}$$

for $R'_{cr} < R < R_c$, and $E_{ex}(R) = 0$ for $R_c \leq R < C$, where the critical rate R'_{cr} has been given in Eq.(39) and the cutoff rate is given by

$$R_c = -\ln\frac{1}{M}\sum_{l=1}^{M}\exp[-4|\alpha|^2\tilde{z}_{(1,l)}], \tag{42}$$

and $\tilde{z}_{(k,l)}$ is given in Eq.(33). □

Note that Eqs.(27) and (41) are identical, because

$$\begin{aligned}\sum_{k=1}^{M}\lambda_k^2 &= \mathrm{Tr}\left(\hat{\rho}(\frac{1}{M},\ldots,\frac{1}{M})\right)^2 \\ &= \mathrm{Sp}\left(\frac{1}{M}\begin{bmatrix}\langle\psi_1|\psi_1\rangle & \cdots & \langle\psi_1|\psi_M\rangle \\ \vdots & \ddots & \vdots \\ \langle\psi_M|\psi_1\rangle & \cdots & \langle\psi_M|\psi_M\rangle\end{bmatrix}\right)^2 \\ &= \frac{1}{M^2}\sum_{k=1}^{M}\sum_{l=1}^{M}|\langle\psi_k|\psi_l\rangle|^2 \\ &= \frac{1}{M^2}\sum_{k=1}^{M}\sum_{l=1}^{M}A_{(k,l)}^2 \\ &= \frac{1}{M^2}\sum_{k=1}^{M}\sum_{l=1}^{M}\exp[-4|\alpha|^2\tilde{z}_{(k,l)}],\end{aligned} \tag{43}$$

where Sp is the trace operation for a matrix.

Finally we consider the optimal distribution yielding the zero-rate reliability function for M-ary PSK coherent state signal. In this case, the function to be maximized is written as

$$\begin{aligned}F_2(\mathbf{p}) &\equiv -\sum_{k=1}^{M}\sum_{l=1}^{M}p_k p_l \ln|\langle\psi_k|\psi_l\rangle|^2 \\ &= 4|\alpha|^2\sum_{k=1}^{M}\sum_{l=1}^{M}p_k p_l \tilde{z}_{(k,l)}.\end{aligned} \tag{44}$$

Then, for any t such that $0 \leq t \leq 1$, we define

$$\begin{aligned}\Delta_{F_2} &\equiv tF_2(\mathbf{p}) + (1-t)F_2(\mathbf{p}') - F_2(t\mathbf{p} + (1-t)\mathbf{p}') \\ &= 4|\alpha|^2 t(1-t)\sum_{k=1}^{M}\sum_{l=1}^{M}q_k q_l \tilde{z}_{(k,l)},\end{aligned} \tag{45}$$

where $\mathbf{q} = \mathbf{p} - \mathbf{p}'$. Since the matrix $\tilde{Z} = [\tilde{z}_{(k,l)}]$ is negative almost-definite, we have $\Delta_{F_2} \leq 0$. Hence $F_2(\mathbf{p})$ is a concave function of $\mathbf{p}$. Therefore, like in the cases treated above, the optimal distribution of the zero-rate reliability function is given by a uniform distribution. Indeed, since the zero-rate reliability function is the limit of the expurgation exponent as $R \to +0$, this result is natural. Consequently we obtain the third proposition.

Proposition 3: The optimal distribution of the zero-rate reliability function for M-ary PSK coherent state signal is given by a uniform distribution on the input alphabet, and the closed-form expression of the zero-rate reliability function is given by $E(+0) = 2|\alpha|^2$. Notice that this expression does not include the size M of the input alphabet. □

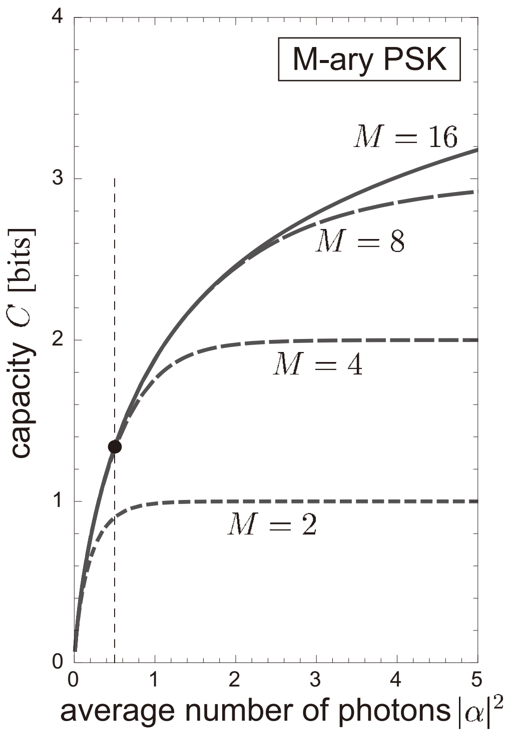

Fig. 3. Channel capacities of 2-, 4-, 8-, and 16-PSK signals.

IV. Example

The aim of this subsection is to illustrate the typical behavior of the error exponents, $E_r(R)$ and $E_{ex}(R)$. With this aim, we take $M = 16$ and $|\alpha|^2 = 0.5$ photons. As shown in Fig. 3, the channel capacity for 16-PSK is $C = 1.3383$ bits (0.92764 nats) per signal. Fig. 4-(a) shows $E_r(R)$ and $E_{ex}(R)$, Fig. 4-(b) shows the optimal value of s for $E_r(R)$, and Fig. 4-(c) shows the optimal value of s for $E_{ex}(R)$. In this case, we obtain the following quantities: the cutoff rate $R_c = 1.1023$ bits (0.76409 nats), the critical rate for the random coding exponent $R_{cr} = 0.95780$ bits (0.66389 nats), and the critical rate for the expurgation exponent $R'_{cr} = 0.30365$ bits (0.21048 nats). The straight line portion of $E_r(R)$ in Fig. 4-(a) is drawn by using Eq.(27). For $0 < R < R_{cr}$, the optimal values of s for $E_r(R)$ are shown as the straight line in Fig. 4-(b) because it always takes $s = 1$ in this region. The curve line portion of $E_r(R)$ in Fig. 4-(a) is drawn by using Eq.(29). At that time, we have carried out numerical computation to find the optimal value of s for $E_r(R)$. The curve line portion in Fig. 4-(b) shows the result of this numerical computation. In the region $0 < R \leq R'_{cr}$, $E_{ex}(R)$ is shown as the curve line in Fig. 4-(a), which is drown by using Eq.(40). In particular, the beginning of the curve line corresponds to $E(+0) = 1.0$ and the end of the curve line touches the straight line portion of $E_r(R)$ at the rate R'_{cr}. The optimal values of s for $E_{ex}(R)$ in the region $0 < R \leq R'_{cr}$ are shown as the curve line in Fig. 4-(c). For $R'_{cr} < R < R_c$, $E_{ex}(R)$ is shown as the straight line in Fig. 4-(a), which passes through the three points $(R'_{cr}, E_r(R'_{cr}))$, $(R_{cr}, E_r(R_{cr}))$, and $(R_c, 0)$. In this region, the optimal value of s for $E_{ex}(R)$ is constant, $s = 1$, which is shown as the straight line in Fig. 4-(c). As mentioned in Section III, $E_r(R)$ and $E_{ex}(R)$ overlap in the region $R'_{cr} \leq R \leq R_{cr}$.

V. Conclusion

We have considered the error exponents for a discrete classical-quantum channel with M-ary PSK coherent state signal. The optimal distributions of the random coding exponent, the expurgation exponent, and the zero-rate reliability function for M-ary PSK coherent state signal have been analytically derived, in which every optimal distribution is given by a uniform distribution on the input alphabet. This result provides the simplified expressions of the error exponents for M-ary PSK coherent state signal. By using our results, we have sketched the error exponents in the case of $M = 16$ to illustrate typical behavior of the error exponents. Our results will be useful for finding good error-correcting codes for quantum communication systems with M-ary PSK coherent state signal.

Appendix

A. $E_x(\mathbf{p}; s)$ is Not a Concave Function of $\mathbf{p}$

Let $M = 3$, $|\alpha|^2 = 1$ and $s = 0.7$ in our system. Taking

$$\mathbf{p} = (0.080808, 0.050505, 0.868687),$$
$$\mathbf{p}' = (0.564356, 0.366336, 0.069308),$$

and $\mathbf{p}'' \equiv (\mathbf{p} + \mathbf{p}')/2$, we have

$$E_x(\mathbf{p}''; s) - \frac{1}{2}\left(E_x(\mathbf{p}; s) + E_x(\mathbf{p}'; s)\right) = 0.323522 > 0. \quad (46)$$

On the other hand, taking

$$\mathbf{p} = (0.134453, 0.126050, 0.739497),$$
$$\mathbf{p}' = (0.012820, 0.012820, 0.974360),$$

we have

$$E_x(\mathbf{p}''; s) - \frac{1}{2}\left(E_x(\mathbf{p}; s) + E_x(\mathbf{p}'; s)\right) = -0.00133881 < 0. \quad (47)$$

Thus, the function $E_x(\mathbf{p}; s)$ is not a concave function of $\mathbf{p}$ for arbitrarily fixed s in general.

B. Optimal Distribution of the Cutoff Rate for M-ary Symmetric Signal

In this paper we have treated the cutoff rate for a discrete classical-quantum channel with pure state signal, mentioned in Eq.(15). The basic properties of the cutoff

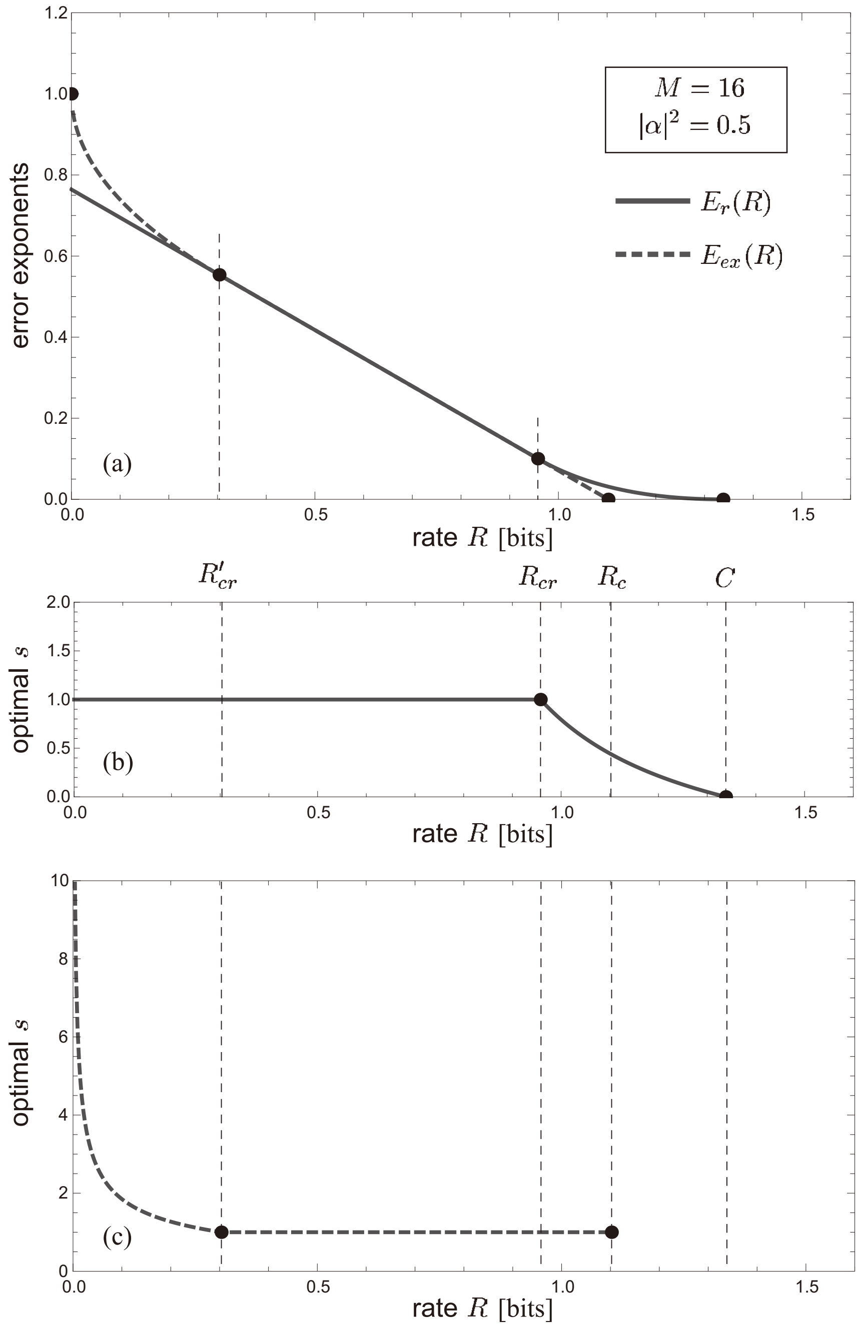

Fig. 4. (a) The random coding exponent, the expurgation exponent, and the zero-rate reliability function for 16-PSK coherent state signal; (b) The optimal s for the random coding exponent; (c) The optimal s for the expurgation exponent.

rate for a discrete classical-quantum channel with symmetric pure state signal have been investigated in detail by Ban *et al.* in Ref.[16]. In particular, they analytically derived the optimal distributions of the cutoff rate and showed its exact expressions in various cases. The result of Eq.(42) is identical to a part of their results, Eq.(52) in Ref.[16].

The cutoff rate is defined not only for a discrete classical-quantum channel with pure state signal but also for a general classical-quantum channel [14]: Let

$$\mathcal{B} = \Big\{\hat{\rho}_1, \hat{\rho}_2, \cdots, \hat{\rho}_M : \hat{\rho}_i \geq 0,\ \mathrm{Tr}\hat{\rho}_i = 1\Big\}. \tag{48}$$

Then, the cutoff rate for a general classical-quantum channel $k \mapsto \hat{\rho}_k$ is defined by

$$R_{c,\mathrm{general}} = -\ln\left[\min_{\mathbf{p}} \Lambda(\mathbf{p})\right], \tag{49}$$

where

$$\Lambda(\mathbf{p}) = \mathrm{Tr}\left(\sum_{k=1}^{M} p_k \sqrt{\hat{\rho}_k}\right)^2. \tag{50}$$

Let us consider the case that the signal satisfies the relation

$$\hat{\rho}_k = \hat{V}^{k-1}\hat{\rho}_1\hat{V}^{\dagger k-1}, \quad k = 1, 2, \cdots, M, \tag{51}$$

with an appropriate unitary operator $\hat{V}$ such that

$$\hat{V}^{\dagger}\hat{V} = \hat{V}\hat{V}^{\dagger} = \hat{1}, \quad \hat{V}^M = \hat{1}. \tag{52}$$

Here we let

$$\hat{\Upsilon}(\mathbf{p}) = \sum_{k=1}^{M} p_k \sqrt{\hat{\rho}_k}. \tag{53}$$

Taking an arbitrary distribution $\mathbf{p} = (p_1, p_2, \cdots, p_M) \equiv \mathbf{p}^{(1)}$ and its permutations

$$\left\{\begin{array}{lcl} \mathbf{p}^{(2)} & = & (p_2, p_3, \cdots, p_1), \\ \mathbf{p}^{(3)} & = & (p_3, p_4, \cdots, p_2), \\ & \vdots & \\ \mathbf{p}^{(M)} & = & (p_M, p_1, \cdots, p_{M-1}), \end{array}\right. \tag{54}$$

we have

$$\sqrt{\hat{\rho}_k} = \hat{V}^{k-1}\sqrt{\hat{\rho}_1}\hat{V}^{\dagger k-1}. \tag{55}$$

Therefore,

$$\hat{\Upsilon}(\mathbf{p}^{(k)}) = \hat{V}^{-(k-1)}\hat{\Upsilon}(\mathbf{p})\hat{V}^{\dagger -(k-1)}. \tag{56}$$

From the convexity of the function $\mathrm{Tr}(\hat{A})^2$,

$$\begin{aligned} \mathrm{Tr}\left(\hat{\Upsilon}(\mathbf{p})\right)^2 &= \frac{1}{M}\sum_{k=1}^{M} \mathrm{Tr}\left(\hat{\Upsilon}(\mathbf{p}^{(k)})\right)^2 \\ &\geq \mathrm{Tr}\left(\sum_{k=1}^{M} \frac{1}{M}\hat{\Upsilon}(\mathbf{p}^{(k)})\right)^2 \\ &= \mathrm{Tr}\left(\hat{\Upsilon}(\frac{1}{M}, \frac{1}{M}, \cdots, \frac{1}{M})\right)^2. \end{aligned} \tag{57}$$

Thus the optimal distribution of the cutoff rate for a symmetric signal defined by Eq.(51) and (52) is given by a uniform distribution.

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