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Transmission of classical information with squeezed states

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***Abstract*—We review studies about Holevo capacity and expurgated bound of reliability function for quantum Gaussian channels. In particular effects of using squeezed states are clarified in terms of such quantum information quantities.**

I. Introduction

We start with recalling early studies on optical communication without the quantum coding theorems. In 1960s the experimental realization of the laser caused outburst of interest in optical communication. Then the ideal laser state was supposed to be a coherent state. Especially Gordon investigated communication systems using coherent states with standard measurements: homodyne and heterodyne detection [15]. Each of these systems was evaluated by Shannon capacity, regarded as a classical memoryless channel. The Shannon capacity for heterodyne and homodyne detections are

$$C_{het} = \log(1+N_{tr}), \quad (1)$$
$$C_{hom} = \frac{1}{2}\log(1+4N_{tr}), \quad (2)$$

where N_{tr} represents the average number of signal photons, called *transmitter energy*. In 1970s Stoler and Yuen introduced the concept of squeezed state (two-photon coherent state) which was expected to be a physical carrier of information more effective than the coherent state [19], [20], [17]. Let a be an annihilation operator. Then a squeezed state is represented as

$$|\mu;\gamma\rangle = \mathcal{D}(\mu)\mathcal{S}(\gamma)|0\rangle, \quad (3)$$

where $\mathcal{D}(\mu) = \exp(\mu a^\dagger - \bar{\mu}a)$ is the displacement operator and $S(\gamma) = \exp((1/2)(-\gamma a^{\dagger 2} + \bar{\gamma}a^2))$ is the squeezing operator with a squeezing parameter γ.

Yuen considered the problem of transmitter quantum state selection from a viewpoint of signal-to-quantum noise ratio under the energy constraint

$$\mathrm{Tr}\rho a^\dagger a \le N_s, \quad (4)$$

where ρ is a density operator representing a quantum state and N_s is a given positive value.

As a result it was found that the optimum signal-to-quantum noise ratio achievable by any state of a radiation field is

$$(S/N)_o = 4N_s(N_s+1), \quad (5)$$

and this value is realized by the squeezed state with the parameter $\gamma = \log\sqrt{1+2N_s}$. Comparing (5) to the value $4N_s$ obtained with coherent states, we can find that the squeezed state is of great advantage to the signal-to-quantum noise ratio. This is why use of squeezed state improves Shannon capacity. Let us consider the optical communication under the transmitter energy constraint N_{tr}, where the transmitter produces squeezed states $|\mu;\gamma_0\rangle$ with the optimum squeezing parameter

$$\gamma_0 = \log\sqrt{1+2N_{tr}}, \quad (6)$$

and the receiver uses a homodyne detection. Then the Shannon capacity is

$$C_{sq} = \log(1+2N_{tr}), \quad (7)$$

which is greater than (1). Here it should be emphasized that (7) is achieved by only one quadrature. Helstrom considered the problem of transmitter quantum state selection for transmission over an attenuation channel. He proved that unless transmitted (input) state is a coherent state, the received (output) state will be a statistical mixture of states [22]. Moreover he compared the performance of squeezed states in binary optical communication through the attenuated channel with that of coherent states. As a result, it was found that the advantage of squeezed state signals over coherent state ones vanishes as the transmittance of channel goes to zero [23].

Besides the performance evaluation for concrete systems, studies of fundamental physical limitations on the quality and rate of information transmission has been carried out by many workers [24], [15], [16], [25]. Forney and Gordon conjectured the entropy bound for information transmission with a fixed set of states valid for arbitrary measurement [24], [16]. However they could not prove the conjecture in the absence of a general theory of measurement process, which was established later by introducing the notion of positive operator valued measurement (POVM) [21]; Gordon showed only the fact that a binary quantum counter can achieve the entropy bound in the limit of the weak signal [15] This fact leads to the notion of binary discretization, which will be explained in Section V. The entropy bound was proved for discrete signal set by Zador [32] and independently by Holevo [1], and for general signal set by Ozawa [18]. In particular, it was found that for a single boson mode under the transmitter energy constraint N_{tr}, the ultimate Shannon capacity is achieved by number states with photon counting as

$$C_{BE} = (N_{tr}+1)\log(N_{tr}+1) - N_{tr}\log N_{tr}, \quad (8)$$

which is called Yuen-Ozawa bound in the following. Communication theory based on quantum channel coding scheme was established in 1990s. Hausladen et al. proved quantum direct coding theorem for channels with finite number of pure states [26]. This theorem claims that the entropy bound proposed by Gordon is achievable in fact by the quantum channel coding, where the measurement can be made over a long sequence of states instead of just symbol by symbol in the sequence. Such measurement is called *entangled measurement*, which is represented by POVM on a Hilbert space $\mathcal{H}^{\otimes n}$, where $\mathcal{H}$ is a Hilbert space providing a quantum-mechanical description for the physical carrier of information and n is the length of sequence of states. The direct coding theorem for discrete mixed states was given by Holevo [5], and Schumacher et al. [27] independently, and was extended to continuous channels with constrained inputs by Holevo [6]. By virtue of these results, it was shown rigorously for the first time that the Yuen-Ozawa bound can be also achieved by coherent states. According to the quantum direct coding theorem, the entropy bound is called Holevo capacity in the following, which is distinguished from Shannon capacity based on the classical channel coding.

A much more detailed and practically applicable description of the asymptotical channel performance than the capacity is given by the reliability function, which is essentially the speed of exponential decay of the error probabilities at information rates below the capacity. The importance of the reliability function is recognized well in classical (Shannon) information theory, and extensive studies have been devoted to it [30]. Quantum coding theorems for the reliability function were established by Holevo. On the analogy from the classical case, Holevo defined the random coding bound $E_r(R)$ and the expurgated bound $E_{ex}(R)$ based on the quantum channel coding, and proved that these give the lower bounds for the reliability function $E(R)$ truly [6]. The random coding bound $E_r(R)$ gives good evaluation of channel performance at high rate, and is defined such that the value of R satisfying $E_r(R) = 0$ is equivalent to the channel capacity. Thus the quantum direct coding theorem can be shown immediately from the random coding bound [6].

On the other hand the expurgated bound $E_{ex}(R)$ is good at low rates. On the analogy of Shannon information theory, we can further consider two important values, the *cut-off rate* and the *zero rate exponents*. The former gives an idea of channel performance at intermediate rates and the latter at low rates. The latter is the value of the reliability function at the zero rate limit and dominates the asymptotic behavior of error probability at low rates.

This paper discusses squeezing effects in terms of Holevo capacity or expurgated bound of quantum reliability function. It is organized as follows. In Section II we introduce Holevo's theory [2] of Gaussian state and give some formulae of quantum information quantities. In Section III we give a general formula of Holevo capacity of classical-quantum Gaussian channel and evaluate squeezing effects for a transmission through an attenuated noisy channel. In Section IV we present formulae of expurgated bound of reliability function and zero rate error exponents and find usage of squeezed states improves those quantum information quantities, which show channel performance at low communication rates. In Section V we introduce an idea of binary discretization.

II. Gaussian State

A. Characteristic Function

Let Z be a real linear space and $\Delta(z,z')$ a nondegenerate bilinear skew symmetric form on Z. Then the pair (Z,Δ) is called a symplectic space. The symplectic space with a nondegenerate bilinear skew symmetric form has the even dimensionality, $\dim Z = 2r$. A basis $\{e_j, h_j\}$ of Z satisfying $\Delta(h_j,e_k) = \delta_{j,k}$, $\Delta(e_j,h_k) = -\delta_{j,k}$,$\Delta(e_j,e_k) = \Delta(h_j,h_k) = 0$ is called *symplectic*, where $\delta_{j,k}$ takes the value of 1 when $j=k$, otherwise 0. For any inner product α, there is a symplectic basis $\{e_j,h_j\}$ in (Z,Δ) in which α takes a standard form

$$\alpha(z,z') = \sum_{j=1}^{r} a_j(x_j x'_j + y_j y'_j),$$

for $z = \sum_{j=1}^{r} x_j e_j + y_j h_j$ and $z' = \sum_{j=1}^{r} x'_j e_j + y'_j h_j$. We call a continuous family of unitary operators $Z \ni z \to V(z)$ satisfying Weyl-Segal relation

$$V(z)V(z') = \mathrm{e}^{i\Delta(z,z')/2}V(z+z'), z, z' \in Z \quad (9)$$

a *representation of the CCR*. For a density operator ρ, we call the transform $z \to \mathcal{F}_z[\rho] = \mathrm{Tr}\rho V(z)$ *characteristic function* of ρ. This is an analogy of classical characteristic function of probability distribution. Let $z \to V(z)$ be irreducible representation of the CCR. Fixing $z \in Z$, we consider a unitary representation $t \to V(tz)$, where from Stone's theorem there exists a self adjoint operator $R(z)$ such that $V(tz) = \exp(itR(z))$. Considering the spectral representation of $R(z)$: $R(z) = \int \lambda E_z(d\lambda)$, we obtain the probability distribution with respect to a state ρ as

$$\mu_\rho^z(d\lambda) = \mathrm{Tr}\rho E_z(d\lambda),$$

whose classical characteristic function is given by the map $t \rightarrow \mathcal{F}_{tz}[\rho]$ as,

$$\mathcal{F}_{tz}[\rho] = \mathrm{Tr}\rho V(tz) = \mathrm{Tr}\rho \int \mathrm{e}^{it\lambda} E_z(d\lambda) = \int \mathrm{e}^{it\lambda} \mathrm{Tr}\rho E_z(d\lambda) = \int \mathrm{e}^{it\lambda} \mu_\rho^z(d\lambda).$$

In classical probability theory, moments of a probability distribution are easily expressed through the derivatives of its characteristic function. An analogous relation exists in quantum case: we denote n^{th} moment of the distribution μ_ρ^z by $m_n(z)$. If n^{th} absolute moment of μ_ρ^z is finite, we have $m_n(z) = i^{-n} \frac{d^n}{dt^n} \mathcal{F}_{tz}[\rho]\Big|_{t=0}$. We call ρ a *state with finite second moments* if $m_2(z) < \infty$ for all $z \in Z$. For a state ρ with finite second moments, we define *mean value* of the state by the function $m(z)$ which is given by 1st moments of classical distribution μ_ρ^z as

$$m(z) = m_1(z),$$

and *correlation function* of the state as

$$\alpha(z, z') := m_2(z, z') - m(z)m(z')$$
$$m_2(z, z') := -\left. \frac{\partial^2}{\partial t \partial s} \mathcal{F}_{tz+sz'}[\rho] \right|_{t=s=0}.$$

Thus like in the classical case we represent the mean values and correlation function by using derivatives of characteristic function of the state. Next we give a rigorous formula of mean value and correlation function. Let $\mathfrak{B}_h(\mathcal{H})$ be an ensemble of Hermitian operator (bounded symmetric operator) on Hilbert space $\mathcal{H}$. We introduce the pre-inner product in $\mathfrak{B}_h(\mathcal{H})$

$$\langle Y, X \rangle_\rho = \mathrm{Tr}\rho(Y \circ X) = \mathrm{Re}\mathrm{Tr}\rho Y X,$$

with $X \circ Y = \frac{1}{2}(XY + YX)$. The completion of $\mathfrak{B}_h(\mathcal{H})$ with respect to $\langle \cdot, \cdot \rangle_\rho$ is a real Hilbert space and it is denoted by $\mathfrak{L}_h^2(\rho)$. Considering $m_2(z) < \infty$ implies $R(z) \in \mathfrak{L}_h^2(\rho)$, we have [2]

$$m(z) = \langle I, R(z) \rangle_\rho$$
$$\alpha(z, z') = \langle R(z) - m(z), R(z') - m(z') \rangle_\rho. \tag{10}$$

In addition the correlation function $\alpha(z, z')$ gives an inner product on Z and satisfies the inequality $\alpha(z, z)\alpha(z', z') \geq \Delta(z, z')^2/4$. Usually $m(z)$ is denoted by $\mathrm{Tr}\rho R(z)$. It may seem to be no problem. But rigorously speaking, $R(z)$ is an unbounded operator and we should be careful whether the trace of $\rho R(z)$ can be defined rigorously. This is why we need to use the inner product $\langle \cdot, \cdot \rangle_\rho$. We can say a similar thing about the correlation function $\alpha(z, z')$.

B. *Definition of Gaussian State*

The Gaussian state is fully characterized by only mean value and correlation function. Let $z \rightarrow V(z)$ be an irreducible representation of the CCR on (Z, Δ). The state ρ is called *Gaussian* if its characteristic function has the form

$$\mathcal{F}_z[\rho] = \exp[im(z) - \frac{1}{2}\alpha(z, z)], \tag{11}$$

where $m(z)$ is a linear functional and $\alpha(z, z')$ is a bilinear symmetric form on Z, which is obtained by Eqs. (10). On the other hand, for (11) to be the characteristic function of a quantum state it is necessary and sufficient that $\alpha(z, z')$ satisfies this uncertainty inequality: $\alpha(z, z)\alpha(z', z') \geq \Delta(z, z')^2/4$.

So far we discussed a general CCR and defined the Gaussian state. In the following we deal with a more concrete CCR. We consider quantum system, such as cavity field with finite numbers of modes, described by annihilation operators $a_1, \ldots, a_r$ satisfying canonical commutation relation (CCR)

$$[a_j, a_k^\dagger] = \delta_{j,k} I, \qquad [a_j, a_k] = 0.$$

The Hilbert space of irreducible representation of this CCR is denoted by $\mathcal{H}$. Let us introduce canonical pairs

$$q_j = \sqrt{\frac{\hbar}{2\omega_j}}(a_j + a_j^\dagger), \quad p_j = i\sqrt{\frac{\hbar\omega_j}{2}}(a_j^\dagger - a_j),$$

satisfying the Heisenberg CCR

$$[q_j, p_k] = i\delta_{jk}\hbar I, \quad [q_j, q_k] = 0, \quad [p_j, p_k] = 0.$$

Using the operators q_j,p_j, we give a representation of the CCR. For a real column $2r$-vector $z = (x_1, y_1, \ldots, x_r, y_r)^T$, we introduce a unitary operators in $\mathcal{H}$ as

$$V(z) = \exp i \sum_{j=1}^{r} (x_j q_j + y_j p_j).$$

Then the operators $V(z)$ satisfy the Weyl-Segal relation (9), where the skew symmetric form Δ is given by

$$\Delta(z, z') = \hbar \sum_{j=1}^{r} (x_j' y_j - x_j y_j') = -z^T \Delta_r z',$$

with the skew symmetric matrix Δ_r. Thus it is found that $V(z)$ gives the representation of the CCR on the symplectic space $(\mathbb{R}^{2r}, \Delta)$.

We can represent $m(z)$ and $\alpha(z, z')$ in Eq. (11) by a mean vector $m = (m_1^q, m_1^p, ..., m_r^q, m_r^p)^T$ and a correlation matrix A as $m(z) = m^T z$ and $\alpha(z, z') = z^T A z'$ respectively.

C. *Examples of Gaussian State*

A single-mode pure Gaussian state is called a *squeezed state*. In particular this paper deals with a squeezed state $|\mu, \gamma\rangle$ with a mean function

$$m(z) = x\sqrt{2\hbar/\omega_1}\mathrm{Re}\mu + y\sqrt{2\hbar\omega_1}\mathrm{Im}\mu \tag{12}$$

and a correlation function

$$\alpha(z, z') = \frac{1}{2}\hbar(xx'e^{-2\gamma}/\omega_1 + \omega_1 e^{2\gamma} yy'), \quad \gamma \in \mathbb{R}. \tag{13}$$

The corresponding mean vector and correlation matrix are given as $m = (\sqrt{2\hbar/\omega_1}\mathrm{Re}\mu, \sqrt{2\hbar\omega_1}\mathrm{Im}\mu)^T$ and

$$A(\gamma) = \begin{bmatrix} \hbar e^{-2\gamma}/2\omega_1 & 0 \\ 0 & \hbar e^{2\gamma}\omega_1/2 \end{bmatrix}, \quad (14)$$

respectively. In the following we put $\omega_1 = 1$ in a single mode case for simplicity.

Another example is a *quasiclassical Gaussian state*

$$\rho_0 = \frac{1}{\pi\bar{N}} \int |\zeta\rangle\langle\zeta| \mathrm{e}^{|\zeta|^2/\bar{N}} d^2\zeta, \quad (15)$$

where mean value is 0 and correlation function is given by

$$\alpha(z, z') = \hbar(\bar{N} + 1/2)(\omega_1^{-1}xx' + \omega_1 yy'),$$

which has the standard form

$$\tilde{\alpha}(z, z') = a(xx' + yy') \quad (16)$$

with $a = \hbar(\bar{N} + 1/2)$. The quasiclassical Gaussian state ρ_0 is represented by the spectral decomposition

$$\rho_0 = \frac{1}{\bar{N}+1} \sum_{n=0}^{\infty} \left(\frac{\bar{N}}{\bar{N}+1}\right)^n |n\rangle\langle n| \quad (17)$$

with number state $|n\rangle$.

D. Basic formulae for Gaussian State

Firstly we obtain the formula of von Neumann entropy for Gaussian states [7]. For simplicity we confine ourselves to the one mode case, which can be easily extended to the multi mode case. Firstly von Neumann entropy of quasiclassical Gaussian state is easily obtained from the spectral decomposition (17) as

$$\begin{aligned} H(\rho_0) =& \frac{\log(\bar{N}+1)}{\bar{N}+1} \sum_{n=0}^{\infty} \left(\frac{\bar{N}}{\bar{N}+1}\right)^n \\ & - \frac{1}{\bar{N}+1} \log \frac{\bar{N}}{\bar{N}+1} \sum_{n=0}^{\infty} n \left(\frac{\bar{N}}{\bar{N}+1}\right)^n \\ =& g(\bar{N}) \end{aligned} \quad (18)$$

with $g(x) = (x+1)\log(x+1) - x\log x$ and $\bar{N} = a/\hbar - 1/2$.

Next we pass to the general one mode case. We consider the representation of the CCR with q_1 and p_1 on symplectic space $(\mathbb{R}^2, \Delta)$. Let ρ be a Gaussian state with a correlation correlation matrix A. Transition from one symplectic basis to another is described by a symplectic transformation S, satisfying $\Delta(Sz, Sz') = \Delta(z, z')$. Let us recall that the correlation function $\alpha(z, z')$ of the Gaussian state ρ takes the standard form by using appropriate symplectic basis. This means that there exists a symplectic transformation S such that

$$SAS^T = \mathrm{diag}[a, a] =: \tilde{A}.$$

Moreover we can find the von Neumann entropy of the Gaussian state ρ is given by $g(\bar{N})$ with $\bar{N} = a/\hbar - 1/2$. We represent the correlation matrix $\tilde{A} = \mathrm{diag}[a, a] = SAS^T$ by the original correlation matrix A without using S. To do that, we shall use the matrix $\Delta_1^{-1}A$ which is diagonalizable and has purely imaginary eigenvalues $\pm ia/\hbar$. For a diagonalizable matrix $M = G\mathrm{diag}(m_j)G^{-1}$, we put $\mathrm{abs}M = G\mathrm{diag}(|m_j|)G^{-1}$. Then considering $\tilde{A}/\hbar = \mathrm{abs}(\Delta_1^{-1}\tilde{A}) = (S^T)^{-1}\mathrm{abs}(\Delta_1^{-1}A)S^T$, we obtain

$$\begin{aligned} H(\rho) = g(\bar{N}) &= g(a/\hbar - \frac{1}{2}) = \frac{1}{2}\mathrm{Sp}g(\tilde{A}/\hbar - I_2) \\ &= \frac{1}{2}\mathrm{Sp}g((S^T)^{-1}\mathrm{abs}(\Delta_1^{-1}A)S^T - I_2) \\ &= \frac{1}{2}\mathrm{Sp}g(\mathrm{abs}(\Delta_1^{-1}A) - I_2/2). \end{aligned} \quad (19)$$

We can easily extend this discussion to the multimode case and obtain the general formula

$$H(\rho) = \frac{1}{2}\mathrm{Sp}g(\mathrm{abs}(\Delta_r^{-1}A) - I_{2r}/2). \quad (20)$$

Similarly by reducing to the quasiclassical Gaussian state with the spectral decomposition (17), we obtain the formula of s-th power of quantum Gaussian state [9]. Let ρ_m be the Gaussian state with correlation matrix α and mean m. Then for any positive real number $s > 0$

$$\mathrm{Tr}\rho_m^s V(z) = \mathcal{N}_s(A) \exp\left[im^T z - \frac{1}{2}z^T \alpha \mathcal{G}_s(A) z\right]. \quad (21)$$

Here the functions $\mathcal{N}_s, \mathcal{G}_s$ are given as

$$\begin{aligned} \mathcal{N}_s(A) &= \left[\det f_s(\mathrm{abs}(\Delta_r^{-1}A))\right]^{-\frac{1}{2}}, \\ \mathcal{G}_s(A) &= g_s(\mathrm{abs}(\Delta_r^{-1}A)), \end{aligned} \quad (22)$$

and

$$\begin{aligned} f_s(d) &= (d+1/2)^s - (d-1/2)^s, \\ g_s(d) &= \frac{1}{2d}\frac{(d+1/2)^2 + (d-1/2)^s}{(d+1/2)^2 - (d-1/2)^s}. \end{aligned} \quad (23)$$

Next we give a formula of product of two Gaussian state [9]. Let ρ_j ($j = 1, 2$) be Gaussian states with characteristic functions

$$\mathrm{Tr}\rho_j V(z) = \exp\left[im_j^T z - \frac{1}{2}z^T A_j z\right]. \quad (24)$$

Using the multiplication formula

$$\begin{aligned} \mathrm{Tr}\rho_1\rho_2 V(z) = \frac{\hbar^\ell}{(2\pi)^\ell} \int & \mathrm{Tr}\rho_1 V(v) \mathrm{Tr}\rho_2 V(z - v) \\ & \cdot \exp\left[\frac{i}{2}v^T \Delta_r z\right] dv, \end{aligned} \quad (25)$$

the characteristic function of $\rho_1\rho_2$ is given as

$$\begin{aligned} \mathrm{Tr}\rho_1\rho_2 V(z) &= [\det \Delta_r^{-1}(A_1 + A_2)]^{-\frac{1}{2}} \\ &\cdot \exp\left[-\frac{1}{2}(m_1 + m_2)^T (A_1 + A_2)^{-1}(m_1 + m_2)\right] \\ &\cdot \exp[-\frac{1}{2}((A_2 + (i/2)\Delta_r)z - 2im_2)^T (A_1 + A_2)^{-1} \\ &\quad ((A_1 - (i/2)\Delta_r)z - 2im_1)]. \end{aligned} \quad (26)$$

Finally a formula of fidelity between two quantum Gaussian states is given as follows [9]. Let ρ_{m1} and ρ_{m2} be Gaussian states with characteristic functions

$$\mathrm{Tr}\rho_{m_j}V(z)=\exp\left[im_j^Tz-\frac{1}{2}z^TAz\right]\qquad(j=1,2). \quad (27)$$

Then the fidelity between these two Gaussian states is

$$\mathrm{Tr}\left|\sqrt{\rho_{m_1}}\sqrt{\rho_{m_2}}\right|=\exp\left[-\frac{1}{8}(m_1-m_2)^TA^{-1}(m_1-m_2)\right]. \quad (28)$$

III. Holevo Capacity of Classical-Quantum Gaussian Channel

A. Holevo Capacity

A classical-quantum Gaussian channel is defined by a mapping $\Theta:\mathbb{R}^{2r}\ni m\to\rho_m\in\mathfrak{S}(\mathcal{H})$, where ρ_m is a quantum Gaussian state with mean function $m(z)=m^Tz$ and correlation function $\alpha(z,z')=z^TAz$. Here m is a $2r$-dim vector representing the functional $m(z)$. Note that ρ_0 describes background noise, comprising quantum noise and ρ_m is obtained by applying the displacement operator to ρ_0. We assume every codeword $(m_1,...,m_n)\in(\mathbb{R}^{2r})^n$ satisfies the energy constraint

$$\sum_{i=1}^{n}f(m_j)\le nE,\quad j=1,...,M,$$

with a energy function $f(m)$. Then the Holevo capacity is given by the expression

$$C=\sup_{\pi\in\mathcal{P}_1}\left[H\left(\int_{\mathbb{R}^{2r}}\rho_m\pi(dm)\right)-\int_{\mathbb{R}^{2r}}H(\rho_m)\pi(dm)\right].$$

where $\mathcal{P}_1$ is the set of Gaussian probability distributions π satisfying

$$\int f(m)\pi(d^2m)<E. \quad (29)$$

The mixture $\rho_\pi=\int\rho_m\pi(dm)$ will be again Gaussian density operator with zero mean and the correlation matrix $A+B$, where A is a correlation matrix of quantum state ρ_0 and B is that of Gaussian probability distribution π. So the Holevo capacity is equal to

$$\begin{aligned}C=&\max_{B\in\mathcal{B}_1}\frac{1}{2}\mathrm{Sp}g(\mathrm{abs}(\Delta_r^{-1}(A+B))-I_{2r}/2)\\&-\frac{1}{2}\mathrm{Sp}g(\mathrm{abs}(\Delta_r^{-1}A)-I_{2r}/2)\end{aligned} \quad (30)$$

where $\mathcal{B}_1$ is the convex set of correlation matrices B corresponding to Gaussian probability distributions satisfying the energy constraint (29).

B. Attenuated Noisy Channel

Let a be an annihilation operator on a Hilbert space $\mathcal{H}$. The linear attenuator with coefficient $k\le1$ is described by the transformation

$$a'=ka+\sqrt{1-k^2}a_0, \quad (31)$$

in the Heisenberg picture. Here a_0 is an annihilation operator in another mode in the Hilbert space $\mathcal{H}_0$ of an "environment". We assume that the environment is initially in the vacuum state. We denote by Γ_{att} the corresponding transformation of states σ in the Shrödinger picture: $\mathrm{Tr}\sigma a'=\mathrm{Tr}\Gamma_{att}[\sigma]a$. Then $\Gamma_{att}[\sigma]$ has the characteristic function [7]

$$\mathrm{Tr}\Gamma_{att}[\sigma]V(z)=\mathrm{Tr}\sigma V(kz)\cdot\exp\left[-\frac{\hbar}{2}\frac{1-k^2}{2}z^Tz\right].$$

Further we assume a thermal noise which zero mean and variance N_c; we denote such defined *attenuated noisy channel* by Γ. Through the attenuated noisy channel we transmit squeezed states $\sigma(\gamma)_m$ with mean vector m and the correlation matrix $A(\gamma)$ given by Eq. (14). Then it can be found that the output state $\rho(\gamma)_m=\Gamma[\sigma(\gamma)_m]$ has the characteristic function

$$\mathrm{Tr}\rho(\gamma)_mV(z)=\mathrm{Tr}\sigma(\gamma)_mV(kz)\cdot\exp\left[-\frac{\hbar}{2}\lambda(k,N_c)z^Tz\right],$$

where

$$\lambda(k,N_c)=\frac{1-k^2}{2}+N_c. \quad (32)$$

This indicates that $\rho(\gamma)_m$ is a Gaussian state with the mean km and the correlation matrix

$$k^2A(\gamma)+\hbar\lambda(k,N_c)\hat{I}. \quad (33)$$

Thus we obtain the classical-quantum channel $m\to\rho(\gamma)_m$ with an energy function

$$f(m)=\frac{1}{2}[m^Tm+\mathrm{Sp}A(\gamma)]. \quad (34)$$

Then the Holevo capacity is given as follows [12].
[A] If $|\gamma|\le\gamma_0$ holds, we obtain

$$\begin{aligned}C=&(k^2N_{tr}+N_c+1)\log(k^2N_{tr}+N_c+1)\\&-(k^2N_{tr}+N_c)\log(k^2N_{tr}+N_c)\\&-g\left(\left\{\left[N_c+\frac{1}{2}\right]^2+2k^2N_{sq}\lambda(k,N_c)\right\}^{1/2}-\frac{1}{2}\right).\end{aligned} \quad (35)$$

where

$$N_{sq}=\mathrm{Tr}\sigma(\gamma)_0a^\dagger=\frac{1}{4}(e^{-2\gamma}+e^{2\gamma})-\frac{1}{2}. \quad (36)$$

When we transmit coherent states, that is $N_{sq}=0$, the second term in (35) is simplified and the capacity is given by

$$\begin{aligned}C=&(k^2N_{tr}+N_c+1)\log(k^2N_{tr}+N_c+1)\\&-(k^2N_{tr}+N_c)\log(k^2N_{tr}+N_c)\\&-(N_c+1)\log(N_c+1)+N_c\log N_c.\end{aligned} \quad (37)$$

[B] When $|\gamma| > \gamma_0$ holds, we obtain

$$C = g\left(\left\{k^2[2N_{tr}+1]\left[\lambda(k,N_c)+k^2\frac{e^{2|\gamma|}}{2}\right] +\lambda(k,N_c)^2 - k^4\frac{e^{4|\gamma|}}{4}\right\}^{1/2} - \frac{1}{2}\right) - g\left(\left\{\left[N_c+\frac{1}{2}\right]^2 + 2k^2N_{sq}\lambda(k,N_c)\right\}^{1/2} - \frac{1}{2}\right). \tag{38}$$

These results indicate usage of coherent states always gives the maximum capacity, and that of squeezed states gives it only for the ideal channel ($k = 1$ and N_c). Furthermore we shall consider the case where the coherent amplitude of the squeezing state is restricted to real number, that is, $m = (m_1^q, 0)$. Then the capacity is calculated as [10]

$$C = g\left(\left[-k^2\frac{e^{2\gamma}}{2}+\frac{1+k^2}{2}+N_c+2k^2N_{tr}\right]^{1/2} \cdot \left[k^2\frac{e^{2\gamma}}{2}+\frac{1-k^2}{2}\right]^{1/2} - \frac{1}{2}\right) - g\left(\left[\left(N_c+\frac{1}{2}\right)^2 + 2k^2N_{sq}\lambda(k,N_c)\right]^{1/2} - \frac{1}{2}\right). \tag{39}$$

In particular, in the ideal case where $k = 1$ and $N_c = 0$, we have

$$C = g\left(\sqrt{\mathcal{B}(\gamma)+\frac{1}{4}} - \frac{1}{2}\right), \tag{40}$$

with

$$\mathcal{B}(\gamma) = e^{2\gamma}\left(2N_{tr}+1\right)/2 - \mathrm{Sp}\alpha(\gamma)/2\hbar = -\frac{1}{4}[e^{2\gamma}-(2N_{tr}+1)]^2 + N_{tr}(N_{tr}+1). \tag{41}$$

We can find $\gamma = \gamma_0$ maximizes $\mathcal{B}(\gamma)$ and also the capacity C. Substituting $\gamma = \gamma_0$ into Eq. (40) we get $C_{BE} = (N_{tr}+1)\log(N_{tr}+1) - N_{tr}\log N_{tr}$, which is equal to the ultimate capacity.

IV. Expurgated Bound of Classical-Quantum Gaussian Channel

A. Definition of Expurgated Bound

Like the classical information theory, we introduce the reliability function

$$E(R) = \limsup_{n\to\infty}\frac{1}{n}\ln\frac{1}{p(e^{nR},n)}, \tag{42}$$

where R is an information rate below the Holevo capacity C and $p(M,n)$ denotes an error probability achieved with the optimal code consisting of M codewords of length n and the optimal quantum detection process described by a positive operator-valued measure. As a lower bound of $E(R)$, the expurgated bound is well known, which gives a good approximation at low rates. It is defined as

$$E_{ex}(R) = \max_{1\le s}(\max_{0\le p}\max_{\pi\in\mathcal{P}_1}\tilde{\mu}(\pi,s,p) - sR), \tag{43}$$

where $\tilde{\mu}$ is a quantum Gallager function given by

$$\tilde{\mu}(\pi,s,p) = -s\ln\int\int e^{p[f(x)+f(y)-2E]} \cdot \left(\mathrm{Tr}\sqrt{\rho_x}\sqrt{\rho_y}\right)^{\frac{1}{s}}\pi(dx)\pi(dy). \tag{44}$$

The quantity characterizing the channel performance at low information rates is the value $E(+0)$ of the reliability function at zero rate. Its lower bound is given by the expurgated bound,

$$E_{ex}(0) \le E(+0), \tag{45}$$

and an upper bound is obtained as [8]

$$E(+0) \le -2\min_{\pi\in\mathcal{P}_1}\int\int \ln\mathrm{Tr}|\sqrt{\rho_m}\sqrt{\rho_{m'}}|\pi(dm)\pi(dm'). \tag{46}$$

In particular, in the case of pure states, the upper and lower bounds coincide, i.e. $E(+0) = E_{ex}(0)$.

The Gallager function $\tilde{\mu}(\pi,s,p)$ with the a priori Gaussian distribution

$$\pi(dm) = \frac{1}{2\pi\sqrt{\det B}}\exp\left[-\frac{1}{2}m^TB^{-1}m\right]dm, \tag{47}$$

takes the following form

$$\tilde{\mu}(\pi,s,p) = 2psE + \frac{s}{2}\log\det\left[(I_2-p\beta)(I_2-pB+2(2s\mathcal{G}_{\frac{1}{s}}(A)A)^{-1}B)\right]. \tag{48}$$

which can be derived from Eqs. (21) and (26).

B. Expurgated Bound for Ideal Channel

When ρ_0 is a coherent state, the expurgated bound can be computed as

$$E_{ex}(R) = \begin{cases} 2N_{tr}(1-\sqrt{1-\mathrm{e}^{-R}}) & R < \log\vartheta(2N_{tr}) \\ 2(N_{tr}+1-\vartheta(2N_{tr})) + \ln\vartheta(2N_{tr}) - R & \text{otherwise} \end{cases}, \tag{49}$$

where

$$\vartheta(x) = \frac{1+\sqrt{x^2+1}}{2}.$$

On the other hand, we have not yet found the way to perform analytically maximization in Eq. (43) and to compute the expurgated bound when ρ_0 is a squeezed state. So we evaluate the expurgated bound for the ideal channel, by considering suboptimal *a priori* distributions to Gaussians with correlation matrix of the form

$$\begin{bmatrix} E & 0 \\ 0 & 0 \end{bmatrix}, \tag{50}$$

with $E = \hbar(2N_{tr}+1) - \mathrm{Sp}A(\gamma)$ for $\gamma \geq 0$. Then we obtain a lower bound $\hat{E}_{ex}(R) (\leq E_{ex}(R) \leq E(R))$ as [14]

$$\hat{E}_{ex}(R) = \begin{cases} 2N_t(N_t+1)(1-\sqrt{1-\mathrm{e}^{-R}}) & R < R_0 \\ \hat{C} - R & \text{otherwise} \end{cases}, \tag{51}$$

with $R_0 = \frac{1}{2}\ln\vartheta(4N_{tr}(N_{tr}+1))$ and

$$\begin{aligned} \hat{C} =& 2N_{tr}(N_{tr}+1) + 1 - \vartheta(4N_{tr}(N_{tr}+1)) \\ &+ \frac{1}{2}\log\vartheta(4N_{tr}(N_{tr}+1)). \end{aligned} \tag{52}$$

Comparing Eq.(49) and Eq.(51), we find that squeezing is good at low communication rates.

C. Zero Rate Error Exponents

The upper and lower bounds of the zero rate exponents for an attenuated noisy channel can be easily obtained as

$$\xi I_0 \leq E(+0) \leq I_0, \tag{53}$$

where

$$\begin{aligned} I_0 &= \frac{k^2\hbar^2}{\lambda^2}\left[N_{sq} + \frac{1}{2} + \sqrt{N_{sq}^2 + N_{sq}}\right][N_{tr} - N_{sq}], \\ \xi &= \frac{1}{1+\sqrt{1-\hbar^2/(4\lambda^2)}}, \\ \lambda &= \hbar\sqrt{(N_c+1/2)^2 + k^2N_{sq}[(1-k^2)+2N_c]}. \end{aligned} \tag{54}$$

In particular, for the ideal channel (i.e. $k = 1$ and $N_c = 0$), we obtain

$$E(+0) = 2\mathcal{B}(\gamma). \tag{55}$$

This shows that the value of zero rate exponents for squeezed states with the optimal parameter $\gamma = \gamma_0$, $2N_{tr}(N_{tr}+1)$, is larger than that for coherent states ($N_{sq} = 0$), $2N_{tr}$. Thus the advantage of squeezing has been shown rigorously by evaluating the zero rate exponents $E(+0)$.

V. Binary Discretization

Let us recall Gordon's suggestion that the binary quantum counter can extract essentially all the information incorporated in a weak light wave; he proposed a semi-optimal system with On-Off keying states $\{|0\rangle, |\alpha\rangle\}$ and the binary quantum counter $\{|0\rangle\langle 0|, I - |0\rangle\langle 0|\}$. According to this suggestion, we infer that the *binary discretization*, restricting the number of letters to only two, realizes asymptotically the capacity in the quantum case. In this section we shall verify this inference by computing the capacity realized by the optimum binary discretization [11]. In addition we also consider the binary discretization for the zero rate exponents.

A. Binary Discretization for Capacity

Let us consider the classical-quantum Gaussian channel with correlation matrix $A(\gamma)$ ($\gamma \geq 0$). This describes an entangled measurement system with pure Gaussian states over the ideal channel Γ_{id} given by the identity operator on $\mathcal{H}$. From (35) and (38) with $k = 1$ and $N_c = 0$, we obtain the capacity of this channel as follows.
[A] When $0 \leq \gamma \leq \gamma_0$ holds, we have

$$C = (N_{tr}+1)\log(N_{tr}+1) - N_{tr}\log N_{tr}, \tag{56}$$

which is denoted by C_{BE}.
[B] When $\gamma > \gamma_0$ holds, we have

$$C = g\left([(2N_{tr}+1)(e^{2\gamma}/2) - e^{4\gamma}/4]^{1/2} - \frac{1}{2}\right). \tag{57}$$

On the other hand the optimum capacity for two signal states is given by

$$C^{(2)} = \sup_{\{m_1,m_2\}}\sup_{Q} H(\{m_1,m_2\},\{Q,1-Q\}). \tag{58}$$

Here the maxima are taken over all binary set of inputs, $\{m_1, m_2\}$, and all probability assignments $Q, 1-Q$ satisfying the constraint

$$Qf(m_1) + (1-Q)f(m_2) \leq \hbar\left(N_{tr} + \frac{1}{2}\right), \tag{59}$$

and $H(\{m_1,m_2\},\{Q,1-Q\})$ is the capacity of binary channel with input letter states $\{\sigma(\gamma)_{m_1}, \sigma(\gamma)_{m_2}\}$ and the corresponding *a priori* probabilities $\{Q, 1-Q\}$. It is known [6] that the capacity $H(\{m_1,m_2\},\{Q,1-Q\})$ is given by

$$\begin{aligned} &H\left(Q\sigma(\gamma)_{m_1} + (1-Q)\sigma(\gamma)_{m_2}\right) \\ &\quad = \log 2 - \frac{1}{2}(1-x)\log(1-x) - \frac{1}{2}(1+x)\log(1+x), \end{aligned} \tag{60}$$

where

$$x = \sqrt{1-4Q(1-Q)(1-\kappa^2)}, \tag{61a}$$

$$\kappa^2 = \mathrm{Tr}\sigma(\gamma)_{m_1}\sigma(\gamma)_{m_2}. \tag{61b}$$

From (26), we have

$$\kappa^2 = \exp\left[-\frac{1}{2\hbar}(m_1-m_2)^T\mathrm{diag}[e^{2\gamma}, e^{-2\gamma}](m_1-m_2)\right]. \tag{62}$$

Carrying out maximization in Eq. (58), we obtain

$$\begin{aligned} C^{(2)} = -\Bigg[&\frac{1-e^{-2\mathcal{B}(\gamma)}}{2}\log\left(\frac{1-e^{-2\mathcal{B}(\gamma)}}{2}\right) \\ &+ \frac{1+e^{-2\mathcal{B}(\gamma)}}{2}\log\left(\frac{1+e^{-2\mathcal{B}(\gamma)}}{2}\right)\Bigg], \end{aligned} \tag{63}$$

which is achieved by $Q = 1/2$, $m_1 = -m_2 = \sqrt{\mathcal{B}(\gamma)}$. Here we can also find a squeezing effect.

The binary discretization realizes approximately the capacity C_{BE} in a weak photon case. Let us consider the case of $\gamma = 0$ for simplicity. By applying $e^{-x} \approx 1 - x$

and $\log(1-x) \approx -x$ to (56) and (63) and neglecting the term of N_{tr}^2, the following approximation holds,

$$C_{BE} \approx C^{(2)} \approx -N_{tr} \log N_{tr} + N_{tr} \qquad \text{for } N_{tr} \ll 1. \quad (64)$$

This result shows that the binary discretization provides a simple optimal strategy to construct a quantum code. The classical continuous channel does not have such a good property[29]; the binary discretization necessarily causes some loss of information and hence it provides no optimal way to use the channel. The code achieving the capacity of continuous channel should be found in more complicated way considering *sphere packing*.

B. Binary Discretization for zero rate exponents

Let us find the optimum binary discretization for zero rate exponents of ideal channel with squeezed states. We consider the optimization,

$$E^{(2)}(+0) = \max_{\{m_1,m_2\}} \max_{Q} E(+0)(\{m_1,m_2\},Q), \quad (65)$$

where $E(+0)(\{m_1,m_2\},Q)$ is the zero rate exponents for the binary channel with two pure states $\sigma(\gamma)_{m_1}$ and $\sigma(\gamma)_{m_2}$ and with *a priori* probability $\{Q, 1-Q\}$. It is known [6] that

$$E(+0)(\{m_1,m_2\},Q) = -2Q(1-Q)\log\kappa^2, \quad (66)$$

where κ^2 is given by (61b). Carrying out the maximization in Eq.(65), we have

$$E^{(2)}(+0) = 2\mathcal{B}(\gamma), \quad (67)$$

This equals to the value of zero rate exponents in the unrestricted case (55).

VI. Conclusion

We have evaluated effects of squeezing for the following cases.

1) capacity for attenuated noisy channel
2) capacity for ideal channel with squeezed state signals whose coherent amplitudes are restricted to real number
3) expurgated bound for ideal channel
4) zero rate exponents for attenuated noisy channel
5) binary discretization for capacity of ideal channel
6) binary discretization for zero rate exponents of ideal channel

In the case of (1), squeezing does not improve capacity; whenever a channel is not ideal the value of capacity decreases by using squeezed states. In the case of (2) and (5), effects of squeezing is so small that it vanishes when transmittance k of channel is small. Only in the case of (3) , (4) and (6) squeezing effects are remarkable. As seen in the subsection V-B, at low communication rates, binary discretization is very effective and hence signal-to-noise ratio is essential. This is the reason why squeezing is good at low communication rates.

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