ISSN 2186-6570

A Lemma on Euler Angles

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Tamagawa University Quantum ICT Research Institute Bulletin, Vol.3, No.1, 25-27, 2013

Tamagawa University Quantum ICT Research Institute Bulletin
Vol.3 No.1 : 25-27 (2013)

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***Abstract*—A recently obtained fundamental lemma on Euler angles and its application to some issue on universal gates for quantum computation [M. Hamada, "Overlooked restrictions on Euler angles in quantum computation," *APS 2013 March Meeting*, Baltimore, USA, 2012, http://meetings.aps.org/link/BAPS.2013.MAR.H1.318 (abstract)] are described in detail. The fundamental lemma in the unpublished work is as follows. Let X, Y, and Z denote the Pauli matrices. For any three-dimensional real unit vector $\hat{n} = (n_x, n_y, n_z)^{\mathrm{T}}$ and $\theta \in \mathbb{R}$, put $R_{\hat{n}}(\theta) = \cos(\theta/2)I - i\sin(\theta/2)(n_x X + n_y Y + n_z Z)$. Put $R_y(\theta) = R_{(0,1,0)^{\mathrm{T}}}(\theta)$ and $R_z(\theta) = R_{(0,0,1)^{\mathrm{T}}}(\theta)$. Lemma: Assume $\alpha, \gamma, \theta \in \mathbb{R}$ and $\hat{n} = (n_x, n_y, n_z)^{\mathrm{T}}$ is a real unit vector; then, there exists some $\beta, \delta \in \mathbb{R}$ satisfying $R_{\hat{n}}(\theta) = e^{i\alpha} R_z(\beta) R_y(\gamma) R_z(\delta)$ if and only if $e^{i\alpha} \in \{1, -1\}$ and $\sqrt{1-n_z^2}|\sin(\theta/2)| = |\sin(\gamma/2)|$. By means of this lemma, a widespread fallacy on universal gates has been pointed out. Mathematical details of the lemma and its application, together with self-contained expositions of underlying notions, are given so as to help us dispel the fallacy.**

I. Introduction

Around quantum computation, many interesting issues have arisen. In particular, it has stimulated investigations on systems represented by finite-dimensional Hilbert spaces and information processing with such systems. One fundamental issue often discussed in the literature on quantum computation is that on realization of an arbitrary unitary operator with universal gates. It is known that, in principle, there exist universal sets of gates, which are building blocks of quantum circuits [1]. Here, a universal set means, using gates in the set, we can construct any unitary operation approximately, and the approximation can be made as accurate as one wants.

Known arguments on the universality reduce the issue of constructing any unitary operation on multiple primitive systems to that of constructing any unitary operation on a single primitive system [2], [3, Sec. 4.5.1], [4, Sec. 3.1]. The primitive system is represented by a two-dimensional Hilbert space, and is sometimes called a quantum-bit system.

This work is related to the issue of constructing an arbitrary unitary operator on the two-dimensional Hilbert space. Its aim is to draw the reader's attention to results of an unpublished piece of work of this author, of which only an abstract was made public [5].

A motivation for that unpublished work was uneasiness about a widespread fallacy often found in textbooks on quantum computation. This fallacy was more than ten years old [3] when it was pointed out by the present author. The fallacy was based on the following erroneous claim. Writing the 'rotation' about a unit vector $\hat{n}$ by an angle θ as $R_{\hat{n}}(\theta)$, they have claimed that any 2×2 unitary matrix can be written as $e^{i\alpha} R_{\hat{m}}(\beta) R_{\hat{n}}(\gamma) R_{\hat{m}}(\delta)$ for appropriate choices of real numbers α, β, γ, and δ if $\hat{n}$ and $\hat{m}$ are non-parallel real unit vectors in three dimensions [3, p. 176, Exercise 4.11], [4, p. 34], [6, p. 66, Theorem 4.2.2].

In that work, the present author has given a fundamental lemma (originally, called a theorem), which shows that there are restrictions on the parameters for an equation $R_{\hat{n}}(\theta) = e^{i\alpha} R_z(\beta) R_y(\gamma) R_z(\delta)$ to hold, where R_z and R_y are special cases of $R_{\hat{n}}$ to be specified below. As an application of this result, it has been shown that the above statement using the non-parallel vectors is incorrect [5].

This work consists of several parts of the unpublished piece of work of this author, the abstract of which is [5]. Specifically, it consists of the fundamental lemma, some part explicating the objects treated in the lemma, and the application of the lemma.

II. Fundamental Lemma on Euler Angles

Let X, Y, and Z denote the Pauli matrices:

$$X = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad Y = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad Z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

Throughout, I denotes the 2×2 identity matrix.

We will work with a matrix

$$R_{\hat{n}}(\theta) = (\cos\tfrac{\theta}{2})I - i(\sin\tfrac{\theta}{2})(n_x X + n_y Y + n_z Z) \quad (1)$$

where $\hat{n} = (n_x, n_y, n_z)^{\mathrm{T}} \in \mathbb{R}^3$ with $\|\hat{n}\| = \sqrt{n_x^2 + n_y^2 + n_z^2} = 1$ and $\theta \in \mathbb{R}$, with $\mathbb{R}$ denoting the set of real numbers. Note in traditional quantum physics [7], special attentions are paid to $R_{\hat{n}}(\theta)$ with $\hat{n} = (0,1,0)^{\mathrm{T}}$ and $R_{\hat{n}}(\theta)$ with $\hat{n} = (0,0,1)^{\mathrm{T}}$, which we denote by $R_y(\theta)$ and $R_z(\theta)$, respectively.

Lemma 1: [5, Theorem]. For any $\alpha, \gamma, \theta \in \mathbb{R}$ and $\hat{n} = (n_x, n_y, n_z)^{\mathrm{T}} \in \mathbb{R}^3$ with $n_x^2 + n_y^2 + n_z^2 = 1$, the following two conditions are equivalent.

I. There exist some $\beta, \delta \in \mathbb{R}$ such that

$$R_{\hat{n}}(\theta) = e^{i\alpha} R_z(\beta) R_y(\gamma) R_z(\delta). \quad (2)$$

II. Both of the following hold:

$$e^{i\alpha} \in \{1, -1\}, \quad (3)$$

$$\sqrt{1-n_z^2}|\sin\tfrac{\theta}{2}| = |\sin\tfrac{\gamma}{2}|. \quad (4)$$

In this article, only a proof of 'I $\Rightarrow$ II' is included.

Proof of Lemma 1 (I $\Rightarrow$ II).

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i) We will first give a more concrete form of (2). A direct calculation shows

$$\begin{aligned} &R_z(\beta)R_y(\gamma)R_z(\delta) \\ &\quad = \cos\frac{\gamma}{2}\cos\frac{\delta+\beta}{2}I - i\sin\frac{\gamma}{2}\sin\frac{\delta-\beta}{2}X \\ &\qquad - i\sin\frac{\gamma}{2}\cos\frac{\delta-\beta}{2}Y - i\cos\frac{\gamma}{2}\sin\frac{\delta+\beta}{2}Z. \quad (5)\end{aligned}$$

Hence, (2) is equivalent to

$$\begin{cases} \cos\frac{\theta}{2} = e^{i\alpha}\cos\frac{\gamma}{2}\cos\frac{\delta+\beta}{2} & (6)\\ n_x\sin\frac{\theta}{2} = e^{i\alpha}\sin\frac{\gamma}{2}\sin\frac{\delta-\beta}{2} & (7)\\ n_y\sin\frac{\theta}{2} = e^{i\alpha}\sin\frac{\gamma}{2}\cos\frac{\delta-\beta}{2} & (8)\\ n_z\sin\frac{\theta}{2} = e^{i\alpha}\cos\frac{\gamma}{2}\sin\frac{\delta+\beta}{2}. & (9)\end{cases}$$

ii) We will prove I $\Rightarrow$ II.

From (6), we have $e^{i\alpha} \in \mathbb{R}$, i.e., (3). On each side of (7) and (8), taking the absolute values, squaring, and summing the resultant pair, we have (4). [Eqs. (6) and (9) also imply (4) similarly.]

□

III. Basics on 2×2 Unitary Matrices

In this section, we derive some basics such as those on the decomposition $R_z(\beta)R_y(\gamma)R_z(\delta)$ and even the matrix $R_{\hat{n}}(\theta)$ itself in an elementary self-contained manner.

A. Rotation about an arbitrary axis

The two matrices $R_z(\theta)$ and $R_y(\theta)$ represent rotations in the following sense. Let $M(x,y,z)$ be defined by

$$M(x,y,z) = xX + yY + zZ = \begin{pmatrix} z & x-iy \\ x+iy & -z \end{pmatrix}$$

for $(x,y,z)^{\mathrm{T}} \in \mathbb{R}^3$. Then, for $\theta \in \mathbb{R}$, we have[1]

$$R_z(\theta)M(x,y,z)R_z(\theta)^\dagger = M(x',y',z')$$

where the coordinates obey

$$(x',y',z')^{\mathrm{T}} = \hat{R}_z(\theta)(x,y,z)^{\mathrm{T}} \quad (10)$$

with

$$\hat{R}_z(\theta) := \begin{pmatrix} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{pmatrix}. \quad (11)$$

We also have

$$R_y(\theta)M(x,y,z)R_y(\theta)^\dagger = M(x',y',z')$$

where

$$(x',y',z')^{\mathrm{T}} = \hat{R}_y(\theta)(x,y,z)^{\mathrm{T}} \quad (12)$$

with

$$\hat{R}_y(\theta) := \begin{pmatrix} \cos\theta & 0 & \sin\theta \\ 0 & 1 & 0 \\ -\sin\theta & 0 & \cos\theta \end{pmatrix}. \quad (13)$$

[1] Considering the action $UM(x,y,z)U^\dagger$ of a unitary matrix U is natural since any 2×2 density matrix can be written in the form $(1/2)[I + M(x,y,z)]$, where $x^2+y^2+z^2 \le 1$.

Thus, $R_z(\theta)$ and $R_y(\theta)$ act as the rotation about the z-axis by the angle θ and the rotation about the y-axis by the angle θ, respectively, in $\mathbb{R}^3$. Using these facts, we will derive a unitary matrix that corresponds to a rotation about an arbitrary vector in $\mathbb{R}^3$.

Now note that any unit vector $\hat{n} \in \mathbb{R}^3$ can be obtained by rotating $(0,0,1)^{\mathrm{T}}$ about the y-axis and then rotating the obtained vector about the z-axis:

$$\hat{n} = \hat{R}_z(\phi)\hat{R}_y(\psi)(0,0,1)^{\mathrm{T}}.$$

As a result, $\hat{n}$ can be written as

$$\hat{n} = (\sin\psi\cos\phi, \sin\psi\sin\phi, \cos\psi)^{\mathrm{T}}, \quad (14)$$

cf. spherical coordinates. Then,

$$\hat{R}_{\hat{n}}(\theta) := \hat{R}_z(\phi)\hat{R}_y(\psi)\hat{R}_z(\theta)[\hat{R}_z(\phi)\hat{R}_y(\psi)]^{-1} \quad (15)$$

is the matrix that represents the rotation about $\hat{n}$ by the angle θ in $\mathbb{R}^3$. This is obvious since $[\hat{R}_z(\phi)\hat{R}_y(\psi)]^{-1}$ moves $\hat{n}$ to $(0,0,1)^{\mathrm{T}}$.

Then,

$$\begin{aligned} U &= R_z(\phi)R_y(\psi)R_z(\theta)[R_z(\phi)R_y(\psi)]^\dagger \\ &= R_z(\phi)R_y(\psi)R_z(\theta)R_y(-\psi)R_z(-\phi) \quad (16)\end{aligned}$$

acts as $UM(x,y,z)U^\dagger = M(x',y',z')$, where $(x',y',z')^{\mathrm{T}} = \hat{R}_{\hat{n}}(\theta)(x,y,z)^{\mathrm{T}}$. Performing the multiplication in (16), we have

$$\begin{aligned} U &= (\cos\tfrac{\theta}{2})I - i(\sin\tfrac{\theta}{2})[(\sin\psi\cos\phi)X \\ &\qquad + (\sin\psi\sin\phi)Y + (\cos\psi)Z], \quad (17)\end{aligned}$$

which is the same as $R_{\hat{n}}(\theta)$ in (1) since we have set $(n_x,n_y,n_z) = (\sin\psi\cos\phi, \sin\psi\sin\phi, \cos\psi)$. Note $R_y(\psi), R_z(\phi), R_z(\theta) \in \mathrm{SU}(2)$, so that $R_{\hat{n}}(\theta) \in \mathrm{SU}(2)$.

B. Parameterizations of SU(2) elements

1) $R_z(\beta)R_y(\gamma)R_z(\delta)$: Any matrix in SU(2) can be written as

$$\begin{pmatrix} e^{-i\eta}\cos\frac{\gamma}{2} & -e^{i\zeta}\sin\frac{\gamma}{2} \\ e^{-i\zeta}\sin\frac{\gamma}{2} & e^{i\eta}\cos\frac{\gamma}{2} \end{pmatrix} \quad (18)$$

and hence, as

$$\begin{pmatrix} e^{-i\frac{\delta+\beta}{2}}\cos\frac{\gamma}{2} & -e^{i\frac{\delta-\beta}{2}}\sin\frac{\gamma}{2} \\ e^{-i\frac{\delta-\beta}{2}}\sin\frac{\gamma}{2} & e^{i\frac{\delta+\beta}{2}}\cos\frac{\gamma}{2} \end{pmatrix} \quad (19)$$

where $\eta, \zeta, \beta, \gamma$, and δ are real numbers. This can be easily shown using the fact that the two columns and the two rows of any unitary matrix are orthonormal, respectively.

Note that the matrix in (19) equals $R_z(\beta)R_y(\gamma)R_z(\delta)$, as can be seen by performing the multiplications. [This can also be written as in (5).] Thus, any matrix in SU(2) can be decomposed into $R_z(\beta)R_y(\gamma)R_z(\delta)$.

2) $R_{\hat{n}}(\theta)$: Any matrix in $\mathrm{SU}(2)$ can be written as

$$\begin{pmatrix} a & b \\ -b^* & a^* \end{pmatrix} \tag{20}$$

with some complex numbers a and b such that $|a|^2+|b|^2=1$. This is just a result of rewriting the expression (18). Hence, any matrix in $\mathrm{SU}(2)$ can be written as

$$\begin{pmatrix} w+iz & y+ix \\ -y+ix & w-iz \end{pmatrix} = wI + i(xX+yY+zZ) \tag{21}$$

with some real numbers x, y, z, and w such that

$$w^2+x^2+y^2+z^2=1. \tag{22}$$

From this expression of an arbitrary matrix in $\mathrm{SU}(2)$, we obtain another parameterization as follows. Take a real number θ such that $\cos(\theta/2)=w$ and $\sin(\theta/2)=\sqrt{1-w^2}=\sqrt{x^2+y^2+z^2}$; write x, y, and z as $x=-n_x\sin(\theta/2), y=-n_y\sin(\theta/2)$, and $z=-n_z\sin(\theta/2)$, where $n_x, n_y, n_z \in \mathbb{R}$ and $n_x^2+n_y^2+n_z^2=1$. Thus, using real numbers $\theta, n_x, n_y, n_z \in \mathbb{R}$ with $n_x^2+n_y^2+n_z^2=1$, any matrix in $\mathrm{SU}(2)$ can be written as

$$(\cos\tfrac{\theta}{2})I - i(\sin\tfrac{\theta}{2})(n_xX+n_yY+n_zZ),$$

which is nothing but $R_{\hat{n}}(\theta)$ in (1). Thus, (17) is another parameterization for elements in $\mathrm{SU}(2)$.

IV. Implication of Lemma 1

In this section, the application of Lemma 1 is presented. The following corollary to Lemma 1 and its consequence are taken from the abstract [5] verbatim, where 'iff' stands for 'if and only if,' and the fallacy is the one mentioned in the introduction.[2]

Corollary 1: Assume $\alpha, \gamma \in \mathbb{R}$, $\hat{n}=(n_x,n_y,n_z)^{\mathrm{T}} \in \mathbb{R}^3$ and $n_x^2+n_y^2+n_z^2=1$. Then, there exist some $\beta, \delta, \theta \in \mathbb{R}$ such that $e^{i\alpha}R_z(\beta)R_{\hat{n}}(\theta)R_z(\delta)=R_y(\gamma)$ iff $e^{i\alpha}=1$ or -1, and $|\cos(\gamma/2)| \geq |n_z|$.

This corollary shows a (to be read 'the' now) widespread fallacy on universal gates in quantum computation. Namely, when $|\cos(\gamma/2)| < |n_z| < 1$, according to a ('the' now) claim often found in textbooks, $R_y(\gamma)$ could be written as $e^{i\alpha}R_z(\beta)R_{\hat{n}}(\theta)R_z(\delta)$ for some $\alpha, \beta, \delta, \theta \in \mathbb{R}$. This is untrue by the corollary.

V. Concluding Remarks

We have drawn the reader's attention to a lemma (theorem in [5]) that clarifies when the two parametric expressions of matrices in $\mathrm{SU}(2)$ equal each other. This lemma was originally obtained to point out the widespread misleading erroneous claim that for any non-parallel vectors $\hat{m}$ and $\hat{n}$, any 2×2 unitary matrix could be written as a scalar multiple of the product of some three $\mathrm{SU}(2)$ rotations about either $\hat{m}$ or $\hat{n}$.

A large portion of the material of the unpublished work, the abstract of which is [5], has been included in more recent conference proceedings [8]. The main result of [8] is obtained with the fundamental theorem of [5] (Lemma 1 of the present work). On the research commenced in [8] (in fact, in the unpublished manuscript, of which [5] is an abstract), some progresses have been made, and obtained results are being presented [9].

[2] We remark that the 'only if' part of the corollary, follows from the 'I $\Rightarrow$ II' part of Lemma 1, and this part is enough for obtaining the consequence, i.e., for demonstrating the fallacy.

Acknowledgments

This work was supported by SCOPE, and by JSPS KAKENHI Grant numbers 22540150 and 21244007.

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[8] M. Hamada, *Proc. FIT 2013*, p. 147, Tottori, Sep. 2013. Corrections: On p. 147, right col, line 14, '$\sqrt{1-\hat{m}^{\mathrm{T}}\hat{n}}$' in the denominator of the fraction should be read '$\sqrt{1-(\hat{m}^{\mathrm{T}}\hat{n})^2}$'; On p. 147, right col, line -11, '$\{\beta \mid (\alpha,\beta,\gamma) \in \mathcal{A}\}$' should be read '$\{\beta \mid \exists\alpha,\gamma \in \mathbb{R},\ (\alpha,\beta,\gamma) \in \mathcal{A}\}$'; On p. 151, right col, line 5, '$U_{\phi,\beta,\gamma}$' should be read '$U_{\alpha,\beta,\gamma}$'.

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