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***Abstract*—The closed-form expression of the square-root measurement for the ternary coherent state signal $\{|0\rangle, |\alpha\rangle, |-\alpha\rangle\}$ is derived. Further, the optimal distribution of the ternary coherent state signal that makes the square-root measurement Bayes-optimal is also derived. From a numerical analysis, it is shown that the average probability of error obtained from the square-root measurement and the corresponding optimal distribution provides a good lower bound of the minimax value for the ternary coherent state signal.**

I. Introduction

Quantum detection theory is one of fundamental tools for evaluating the performance limit of optical digital communication systems [1], [2]. When the probability distribution of quantum state signals to be transmitted from the sender is unknown at all, a better approach for finding the minimum average probability of error of the communication system under consideration is to employ the quantum minimax strategy [3], [4], [5] rather than the quantum Bayes strategy. The minimum average probability of error in terms of the quantum minimax strategy is called the minimax value. In general, analytical derivation of the minimax value for a given set of quantum state signals is difficult. Although one can use a numerical calculation procedure [6], finding the minimax value with a required computational accuracy will be not easy when the number of quantum state signals becomes quite large. This motivates us to find an appropriate quantity that behaves like an alternative to the minimax value and is easy to calculate. For this purpose, we focus attention on the square-root measurement [7], [8], [9] because of its simple structure. According to the literature [10], the square-root measurement for distinguishing linearly independent pure state signals becomes Bayes-optimal with certain probability distribution that is given by the diagonal entries of the square-root of the Gram matrix formed by the state vectors of the signals. From the definition of the minimax value, the average probability of error obtained by the square-root measurement for linearly independent pure state signals and the corresponding optimal probability distribution provides a lower bound for the minimax value in general.

In this study, we are aiming to investigate the tightness of this lower bound for the minimax value. As the first step to this aim, the square-root measurement for ternary coherent state signal $\{|0\rangle, |\alpha\rangle, |-\alpha\rangle\}$ is considered in this paper. In the context of the advanced optical modulation formats for fiber-optic communication systems [11], this type of ternary signal is actually used in some modulation techniques such as duobinary and alternate mark inversion. Moreover, in the context of the optical reading schemes or optical radar systems that are operated with binary phase shift keying signal, this type of ternary signal describes a situation that reflected signals from the target randomly disappeared on the way to the receiver. Taking the difficulty of estimation of the signal disappearance rate into account, the minimax approach to this ternary signal is suitable for the right evaluation of its error performance. Thus the error performance analysis of the ternary coherent state signal itself is relevant to our interest in developing design theory for practical quantum communication systems, as well as the discussion on the tightness of the lower bound for the minimax value of it.

The remaining part of this paper is organized as follows. The column vector representation of the ternary coherent state signal and the closed-form expression of the corresponding square-root measurement are shown in Section II, and the minimax case is reviewed in Section III. Using the results of Sections II and III, numerical comparison between the minimax value and the lower bound obtained from the square-root measurement is performed in Section IV, together with other two cases that the uniform distribution of the signal is supposed, and we summarize this paper in Section V.

II. Square-Root Measurement for Ternary Coherent State Signal

Let $\mathcal{A} = \{0, +, -\}$ be the alphabet of signaling symbols. For each symbol, we set

$$\text{“}0\text{”} \quad \mapsto \quad |\psi_0\rangle = |0\rangle, \tag{1}$$
$$\text{“}+\text{”} \quad \mapsto \quad |\psi_+\rangle = |\alpha\rangle, \tag{2}$$
$$\text{“}-\text{”} \quad \mapsto \quad |\psi_-\rangle = |-\alpha\rangle, \tag{3}$$

where $|0\rangle$ is the vacuum state, and $|\pm\alpha\rangle$ are the coherent states of light having amplitudes $\alpha > 0$ and $-\alpha$, respectively. Here we define

$$
\begin{aligned}
|\phi_1\rangle &= c_{11}|\psi_0\rangle + c_{12}|\psi_+\rangle + c_{13}|\psi_-\rangle, && (4)\\
|\phi_2\rangle &= c_{21}|\psi_0\rangle + c_{22}|\psi_+\rangle + c_{23}|\psi_-\rangle, && (5)\\
|\phi_3\rangle &= c_{31}|\psi_0\rangle + c_{32}|\psi_+\rangle + c_{33}|\psi_-\rangle, && (6)
\end{aligned}
$$

with the coefficients

$$
\left\{
\begin{aligned}
c_{11} &= 0,\\
c_{12} = c_{13} &= \frac{1}{\sqrt{2(1+\kappa^4)}},\\
c_{21} &= 0,\\
c_{22} &= \frac{1}{\sqrt{2(1-\kappa^4)}},\\
c_{23} &= -\frac{1}{\sqrt{2(1-\kappa^4)}},\\
c_{31} &= \frac{\sqrt{1+\kappa^4}}{1-\kappa^2},\\
c_{32} = c_{33} &= -\frac{\kappa}{(1-\kappa^2)\sqrt{1+\kappa^4}},
\end{aligned}
\right. \tag{7}
$$

where $\kappa = \exp[-|\alpha|^2/2]$. These vectors form an ordered orthonormal basis, $\gamma = \{|\phi_1\rangle, |\phi_2\rangle, |\phi_3\rangle\}$. By using this orthonormal basis γ, the signal states can be represented as

$$
|\psi_0\rangle \doteq [|\psi_0\rangle]_\gamma = \begin{bmatrix} w \\ 0 \\ x \end{bmatrix}, \tag{8}
$$

$$
|\psi_+\rangle \doteq [|\psi_+\rangle]_\gamma = \begin{bmatrix} y \\ +z \\ 0 \end{bmatrix}, \tag{9}
$$

$$
|\psi_-\rangle \doteq [|\psi_-\rangle]_\gamma = \begin{bmatrix} y \\ -z \\ 0 \end{bmatrix}, \tag{10}
$$

where the entries are given by

$$
\left\{
\begin{aligned}
w &= \frac{\sqrt{2}\kappa}{\sqrt{1+\kappa^4}},\\
x &= \frac{1-\kappa^2}{\sqrt{1+\kappa^4}},\\
y &= \frac{\sqrt{1+\kappa^4}}{\sqrt{2}},\\
z &= \frac{\sqrt{1-\kappa^4}}{\sqrt{2}}.
\end{aligned}
\right. \tag{11}
$$

For the signal set $\{|\psi_0\rangle, |\psi_+\rangle, |\psi_-\rangle\}$, the decision vectors of the square-root measurement are written as

$$
\begin{aligned}
|d_0^\bullet\rangle &= \hat{G}^{-1/2}|\psi_0\rangle, && (12)\\
|d_+^\bullet\rangle &= \hat{G}^{-1/2}|\psi_+\rangle, && (13)\\
|d_-^\bullet\rangle &= \hat{G}^{-1/2}|\psi_-\rangle, && (14)
\end{aligned}
$$

where the Gram operator $\hat{G}$ is given by $\hat{G} = |\psi_0\rangle\langle\psi_0| + |\psi_+\rangle\langle\psi_+| + |\psi_-\rangle\langle\psi_-|$. Since the signal set is linearly independent, the vectors $|d_0^\bullet\rangle$, $|d_+^\bullet\rangle$, and $|d_-^\bullet\rangle$ are orthonormal. Therefore, the decision operators of the square-root measurement, $\hat{\Pi}_0^\bullet = |d_0^\bullet\rangle\langle d_0^\bullet|$, $\hat{\Pi}_+^\bullet = |d_+^\bullet\rangle\langle d_+^\bullet|$, and $\hat{\Pi}_-^\bullet = |d_-^\bullet\rangle\langle d_-^\bullet|$, form a projection-valued measure (PVM). The Gram matrix of the ternary coherent state signal is

$$
\begin{aligned}
G &= \begin{bmatrix} \langle\psi_0|\psi_0\rangle & \langle\psi_0|\psi_+\rangle & \langle\psi_0|\psi_-\rangle \\ \langle\psi_+|\psi_0\rangle & \langle\psi_+|\psi_+\rangle & \langle\psi_+|\psi_-\rangle \\ \langle\psi_-|\psi_0\rangle & \langle\psi_-|\psi_+\rangle & \langle\psi_-|\psi_-\rangle \end{bmatrix}\\
&= \begin{bmatrix} 1 & \kappa & \kappa \\ \kappa & 1 & \kappa^4 \\ \kappa & \kappa^4 & 1 \end{bmatrix}.
\end{aligned} \tag{15}
$$

The eigenvalues of G are given by

$$
\begin{aligned}
\lambda_1 &= 1-\kappa^4, && (16)\\
\lambda_2 &= \frac{1}{2}\left(2+\kappa^4-\kappa\sqrt{8+\kappa^6}\right), && (17)\\
\lambda_3 &= \frac{1}{2}\left(2+\kappa^4+\kappa\sqrt{8+\kappa^6}\right), && (18)
\end{aligned}
$$

and the eigenvectors are given by

$$
\vec{\lambda}_1 = \begin{bmatrix} \lambda_1^{(1)} \\ \lambda_1^{(2)} \\ \lambda_1^{(3)} \end{bmatrix} \quad \text{(corresponding to } \lambda_1\text{)}, \tag{19}
$$

$$
\vec{\lambda}_2 = \begin{bmatrix} \lambda_2^{(1)} \\ \lambda_2^{(2)} \\ \lambda_2^{(3)} \end{bmatrix} \quad \text{(corresponding to } \lambda_2\text{)}, \tag{20}
$$

$$
\vec{\lambda}_3 = \begin{bmatrix} \lambda_3^{(1)} \\ \lambda_3^{(2)} \\ \lambda_3^{(3)} \end{bmatrix} \quad \text{(corresponding to } \lambda_3\text{)} \tag{21}
$$

with

$$
\left\{
\begin{aligned}
\lambda_1^{(1)} &= 0,\\
\lambda_1^{(2)} &= -\frac{1}{\sqrt{2}},\\
\lambda_1^{(3)} &= \frac{1}{\sqrt{2}},\\
\lambda_2^{(1)} &= -\frac{1}{\sqrt{2}} \cdot \frac{\kappa^3+\sqrt{8+\kappa^6}}{\sqrt{8+\kappa^6+\kappa^3\sqrt{8+\kappa^6}}},\\
\lambda_2^{(2)} &= \frac{1}{\sqrt{2}} \cdot \frac{2}{\sqrt{8+\kappa^6+\kappa^3\sqrt{8+\kappa^6}}},\\
\lambda_2^{(3)} &= \lambda_2^{(2)}\\
\lambda_3^{(1)} &= \frac{1}{\sqrt{2}} \cdot \frac{-\kappa^3+\sqrt{8+\kappa^6}}{\sqrt{8+\kappa^6-\kappa^3\sqrt{8+\kappa^6}}},\\
\lambda_3^{(2)} &= \frac{1}{\sqrt{2}} \cdot \frac{2}{\sqrt{8+\kappa^6-\kappa^3\sqrt{8+\kappa^6}}},\\
\lambda_3^{(3)} &= \lambda_3^{(2)}
\end{aligned}
\right. \tag{22}
$$

where the eigenvectors have been normalized. Using these eigenvalues and eigenvectors, we have the square-root of the Gram matrix as follows:

$$
G^{1/2} = \begin{bmatrix} f_{11} & f_{12} & f_{13} \\ f_{12} & f_{22} & f_{23} \\ f_{13} & f_{23} & f_{33} \end{bmatrix} \tag{23}
$$

with

$$\begin{cases} f_{11} &=& \dfrac{2-\kappa^2}{\sqrt{4-2\kappa^2+\kappa^4}}, \\ f_{12} &=& \dfrac{\kappa}{\sqrt{4-2\kappa^2+\kappa^4}}, \\ f_{13} &=& f_{12}, \\ f_{22} &=& \dfrac{2-\kappa^2+\kappa^4+\sqrt{1-\kappa^4}\sqrt{4-2\kappa^2+\kappa^4}}{2\sqrt{4-2\kappa^2+\kappa^4}}, \\ f_{23} &=& \dfrac{2-\kappa^2+\kappa^4-\sqrt{1-\kappa^4}\sqrt{4-2\kappa^2+\kappa^4}}{2\sqrt{4-2\kappa^2+\kappa^4}}, \\ f_{33} &=& f_{22}. \end{cases} \tag{24}$$

Further, the inverse of the square-root of the Gram matrix is given by

$$G^{-1/2} = \begin{bmatrix} h_{11} & h_{12} & h_{13} \\ h_{12} & h_{22} & h_{23} \\ h_{13} & h_{23} & h_{33} \end{bmatrix} \tag{25}$$

with

$$\begin{cases} h_{11} &=& \dfrac{2-\kappa^2+\kappa^4}{(1-\kappa^2)\sqrt{4-2\kappa^2+\kappa^4}}, \\ h_{12} &=& -\dfrac{\kappa}{(1-\kappa^2)\sqrt{4-2\kappa^2+\kappa^4}}, \\ h_{13} &=& h_{12} \\ h_{22} &=& \dfrac{2+\kappa^2-\kappa^4+\sqrt{1-\kappa^4}\sqrt{4-2\kappa^2+\kappa^4}}{2(1-\kappa^4)\sqrt{4-2\kappa^2+\kappa^4}}, \\ h_{23} &=& -\dfrac{2+\kappa^2-\kappa^4+\sqrt{1-\kappa^4}\sqrt{4-2\kappa^2+\kappa^4}}{2(1-\kappa^4)\sqrt{4-2\kappa^2+\kappa^4}}, \\ h_{33} &=& h_{22}. \end{cases} \tag{26}$$

Then the decision vectors of the square-root measurement are expressed in the following form:

$$|d_0^\bullet\rangle \doteq [|d_0^\bullet\rangle]_\gamma = \begin{bmatrix} s \\ 0 \\ t \end{bmatrix}, \tag{27}$$

$$|d_+^\bullet\rangle \doteq [|d_+^\bullet\rangle]_\gamma = \frac{1}{\sqrt{2}}\begin{bmatrix} t \\ 1 \\ -s \end{bmatrix}, \tag{28}$$

$$|d_-^\bullet\rangle \doteq [|d_-^\bullet\rangle]_\gamma = \frac{1}{\sqrt{2}}\begin{bmatrix} t \\ -1 \\ -s \end{bmatrix}, \tag{29}$$

where

$$s = \frac{\sqrt{2}\kappa}{\sqrt{1+\kappa^4}\sqrt{4-2\kappa^2+\kappa^4}}, \tag{30}$$

$$t = \frac{2-\kappa^2+\kappa^4}{\sqrt{1+\kappa^4}\sqrt{4-2\kappa^2+\kappa^4}}. \tag{31}$$

Now we set $\mathbf{p}^\bullet = (p_0^\bullet, p_+^\bullet, p_-^\bullet)$ by

$$\begin{cases} p_0^\bullet &=& \dfrac{2-\kappa^2+\kappa^4+\sqrt{1-\kappa^4}\sqrt{4-2\kappa^2+\kappa^4}}{10-5\kappa^2+\kappa^4+\sqrt{1-\kappa^4}\sqrt{4-2\kappa^2+\kappa^4}}, \\ p_+^\bullet &=& \dfrac{4-2\kappa^2}{10-5\kappa^2+\kappa^4+\sqrt{1-\kappa^4}\sqrt{4-2\kappa^2+\kappa^4}}, \\ p_-^\bullet &=& p_+^\bullet. \end{cases} \tag{32}$$

It is easy to verify that $p_a^\bullet > 0$, $a \in \mathcal{A}$, and $p_0^\bullet + p_+^\bullet + p_-^\bullet = 1$; that is, $\mathbf{p}^\bullet$ is a probability distribution. With this probability distribution, the square-root measurement $\Pi^\bullet$ satisfies the Bayes-optimality conditions. Therefore the minimum average probability of error at $\mathbf{p}^\bullet$ is given as follows.

$$\begin{aligned} \bar{P}_\mathrm{e}^\bullet &= \min_{\Pi \in \mathcal{D}} \bar{P}_\mathrm{e}(\mathbf{p}^\bullet, \Pi) \\ &= \bar{P}_\mathrm{e}(\mathbf{p}^\bullet, \Pi^\bullet) \\ &= 1 - p_0^\bullet P_{0|0}^\bullet - p_+^\bullet P_{+|+}^\bullet - p_-^\bullet P_{-|-}^\bullet \\ &= \Big\{ 4\sqrt{4-2\kappa^2+\kappa^4} \\ &\qquad -2\left(4-5\kappa^2+3\kappa^4-\kappa^6\right)\sqrt{1-\kappa^4} \Big\} \\ &\quad / \Big\{ \left(10-5\kappa^2+\kappa^4\right)\sqrt{4-2\kappa^2+\kappa^4} \\ &\qquad + \left(4-2\kappa^2+\kappa^4\right)\sqrt{1-\kappa^4} \Big\}, \end{aligned} \tag{33}$$

where the conditional probabilities are given by

$$P_{0|0}^\bullet = f_{11}^2 = \frac{4-4\kappa^2+\kappa^4}{4-2\kappa^2+\kappa^4}, \tag{34}$$

$$\begin{aligned} P_{+|+}^\bullet &= f_{22}^2 \\ &= \frac{(2-\kappa^2+\kappa^4)\sqrt{1-\kappa^4}\sqrt{4-2\kappa^2+\kappa^4}}{2\left(4-2\kappa^2+\kappa^4\right)} \\ &\quad + \frac{4-3\kappa^2+\kappa^4}{2\left(4-2\kappa^2+\kappa^4\right)}, \end{aligned} \tag{35}$$

$$P_{-|-}^\bullet = f_{33}^2 = P_{+|+}^\bullet. \tag{36}$$

Observe that the probabilities $p_a^\bullet$ are rewritten as

$$\begin{cases} p_0^\bullet &=& Z/f_{11}, \\ p_+^\bullet &=& Z/f_{22}, \\ p_-^\bullet &=& Z/f_{33}, \end{cases} \tag{37}$$

where

$$\begin{aligned} Z &= \frac{2-\kappa^2+\kappa^4+\sqrt{1-\kappa^4}\sqrt{4-2\kappa^2+\kappa^4}}{10-5\kappa^2+\kappa^4+\sqrt{1-\kappa^4}\sqrt{4-2\kappa^2+\kappa^4}} \\ &\quad \times \frac{2-\kappa^2}{\sqrt{4-2\kappa^2+\kappa^4}}. \end{aligned} \tag{38}$$

Thus, this result is an example for Theorem 5 of [10].

III. Minimax Discrimination

Here we mention the minimax discrimination for the ternary coherent state signal in briefly. Let $\Pi^\circ = (\hat{\Pi}_0^\circ, \hat{\Pi}_+^\circ, \hat{\Pi}_-^\circ) = (|d_0^\circ\rangle\langle d_0^\circ|, |d_+^\circ\rangle\langle d_+^\circ|, |d_-^\circ\rangle\langle d_-^\circ|)$ denote the minimax POVM for the ternary coherent state signal. The minimax decision vectors are given [6] by

$$|d_0^\circ\rangle \doteq [|d_0^\circ\rangle]_\gamma = \begin{bmatrix} u \\ 0 \\ v \end{bmatrix}, \tag{39}$$

$$|d_+^\circ\rangle \doteq [|d_+^\circ\rangle]_\gamma = \frac{1}{\sqrt{2}}\begin{bmatrix} v \\ 1 \\ -u \end{bmatrix}, \tag{40}$$

$$|d_-^\circ\rangle \doteq [|d_-^\circ\rangle]_\gamma = \frac{1}{\sqrt{2}}\begin{bmatrix} v \\ -1 \\ -u \end{bmatrix}, \tag{41}$$

where

$$u = \sqrt{2}\left(\frac{2\kappa\sqrt{1-\kappa^4}}{(1+4\kappa^2+\kappa^4)\sqrt{1+\kappa^4}} - \frac{(1-2\kappa^2-\kappa^4)\kappa\sqrt{2+\kappa^2}}{(1+4\kappa^2+\kappa^4)\sqrt{1+\kappa^4}}\right), \quad (42)$$

$$v = \frac{(1-2\kappa^2-\kappa^4)\sqrt{1-\kappa^4}}{(1+4\kappa^2+\kappa^4)\sqrt{1+\kappa^4}} + \frac{4\kappa^2\sqrt{2+\kappa^2}}{(1+4\kappa^2+\kappa^4)\sqrt{1+\kappa^4}}. \quad (43)$$

The minimax distribution $\mathbf{p}^\circ = (p_0^\circ, p_+^\circ, p_-^\circ)$ is given by

$$p_0^\circ = \frac{2\sqrt{1-\kappa^4} - (1-2\kappa^2-\kappa^4)\sqrt{2+\kappa^2}}{(1+4\kappa^2+\kappa^4)\sqrt{2+\kappa^2}}, \quad (44)$$

$$p_+^\circ = \frac{-\sqrt{1-\kappa^4} + (1+\kappa^2)\sqrt{2+\kappa^2}}{(1+4\kappa^2+\kappa^4)\sqrt{2+\kappa^2}}, \quad (45)$$

$$p_-^\circ = p_+^\circ. \quad (46)$$

Further, the minimax value $\bar{P}_\mathrm{e}^\circ$ is given by

$$\begin{aligned}\bar{P}_\mathrm{e}^\circ &= \min_{\Pi\in\mathcal{D}} \max_{\mathbf{p}\in\mathcal{P}} \bar{P}_\mathrm{e}(\Pi,\mathbf{p}) \\ &= \bar{P}_\mathrm{e}(\Pi^\circ,\mathbf{p}^\circ) \\ &= \left(3+2\kappa^2+\kappa^4-2\sqrt{2+\kappa^2}\sqrt{1-\kappa^4}\right) \\ &\quad \times \frac{2\kappa^2\left(1+\kappa^2\right)}{(1+4\kappa^2+\kappa^4)^2}. \quad (47)\end{aligned}$$

From the definition of the minimax strategy, it is clear that $\bar{P}_\mathrm{e}^\circ \geq \bar{P}_\mathrm{e}^\bullet$.

IV. Numerical Analysis

To begin with, we compare the minimax value $\bar{P}_\mathrm{e}^\circ$ with the minimum average probability $\bar{P}_\mathrm{e}^\bullet$ of error at $\mathbf{p}^\bullet$, the average probability $\bar{P}_\mathrm{e}(\Pi^\bullet,\mathbf{u})$ of error that is obtained from the square-root measurement $\Pi^\bullet$ and the uniform distribution $\mathbf{u} = (1/3, 1/3, 1/3)$, and the minimum average probability $\bar{P}_\mathrm{e}^\mathrm{bayes}(\mathbf{u})$ of error at $\mathbf{u}$. TABLE I shows the numerical calculation results for $\kappa = 0.9$, 0.7, 0.5, 0.3, and 0.1. From this table, we see that $\bar{P}_\mathrm{e}^\bullet$ provides a good lower bound of the minimax value $\bar{P}_\mathrm{e}^\circ$. Further, we observe

$$\bar{P}_\mathrm{e}^\circ > \bar{P}_\mathrm{e}^\bullet > \bar{P}_\mathrm{e}(\Pi^\bullet,\mathbf{u}) \gtrsim \bar{P}_\mathrm{e}^\mathrm{bayes}(\mathbf{u}) \quad (48)$$

for $0 < \kappa < 1$. The left-most inequality is due to the fact that $\mathbf{p}^\circ \neq \mathbf{p}^\bullet$ for $0 < \kappa < 1$, and the right-most inequality reflects the result of the literature [8]; when κ is very small, then $G \sim I$ and hence $\bar{P}_\mathrm{e}(\Pi^\bullet,\mathbf{u}) \sim \bar{P}_\mathrm{e}^\mathrm{bayes}(\mathbf{u})$.

Next, we take $\kappa = 0.5$ to understand the relation (48) graphically. In this case, we have the following numerical

TABLE I
$\bar{P}_\mathrm{e}^\circ$, $\bar{P}_\mathrm{e}^\bullet$, $\bar{P}_\mathrm{e}(\Pi^\bullet,\mathbf{u})$, and $\bar{P}_\mathrm{e}^\mathrm{bayes}(\mathbf{u})$

Parameter		Probability of error	
κ	$= 0.9$	$\bar{P}_\mathrm{e}^\circ$	$= 0.404879$
$\|\alpha\|^2$	$= 0.210721$	$\bar{P}_\mathrm{e}^\bullet$	$= 0.402008$
		$\bar{P}_\mathrm{e}(\Pi^\bullet,\mathbf{u})$	$= 0.393017$
		$\bar{P}_\mathrm{e}^\mathrm{bayes}(\mathbf{u})$	$= 0.386295$
κ	$= 0.7$	$\bar{P}_\mathrm{e}^\circ$	$= 0.209461$
$\|\alpha\|^2$	$= 0.713350$	$\bar{P}_\mathrm{e}^\bullet$	$= 0.205191$
		$\bar{P}_\mathrm{e}(\Pi^\bullet,\mathbf{u})$	$= 0.201989$
		$\bar{P}_\mathrm{e}^\mathrm{bayes}(\mathbf{u})$	$= 0.201362$
κ	$= 0.5$	$\bar{P}_\mathrm{e}^\circ$	$= 0.0966410$
$\|\alpha\|^2$	$= 1.38629$	$\bar{P}_\mathrm{e}^\bullet$	$= 0.0941934$
		$\bar{P}_\mathrm{e}(\Pi^\bullet,\mathbf{u})$	$= 0.0935778$
		$\bar{P}_\mathrm{e}^\mathrm{bayes}(\mathbf{u})$	$= 0.0935369$
κ	$= 0.3$	$\bar{P}_\mathrm{e}^\circ$	$= 0.0323350$
$\|\alpha\|^2$	$= 2.40795$	$\bar{P}_\mathrm{e}^\bullet$	$= 0.0314511$
		$\bar{P}_\mathrm{e}(\Pi^\bullet,\mathbf{u})$	$= 0.0313877$
		$\bar{P}_\mathrm{e}^\mathrm{bayes}(\mathbf{u})$	$= 0.0313865$
κ	$= 0.1$	$\bar{P}_\mathrm{e}^\circ$	$= 0.00344978$
$\|\alpha\|^2$	$= 4.60517$	$\bar{P}_\mathrm{e}^\bullet$	$= 0.00335168$
		$\bar{P}_\mathrm{e}(\Pi^\bullet,\mathbf{u})$	$= 0.00335097$
		$\bar{P}_\mathrm{e}^\mathrm{bayes}(\mathbf{u})$	$= 0.00335097$

calculation results:

$$\begin{aligned}
|\psi_0\rangle &\doteq {}^t[\ 0.685994,\ 0.000000,\ 0.727607],\\
|\psi_+\rangle &\doteq {}^t[\ 0.728869,\ 0.684653,\ 0.000000],\\
|\psi_-\rangle &\doteq {}^t[\ 0.728869, -0.684653,\ 0.000000],\\
\mathbf{p}^\circ &= (\ 0.413815,\ 0.293092,\ 0.293092);\\
|d_0^\circ\rangle &\doteq {}^t[\ 0.425813,\ 0.000000,\ 0.904811],\\
|d_+^\circ\rangle &\doteq {}^t[\ 0.639798,\ 0.707107, -0.301095],\\
|d_-^\circ\rangle &\doteq {}^t[\ 0.639798, -0.707107, -0.301095];\\
\bar{P}_\mathrm{e}^\circ &= 0.0966410;\\
\mathbf{p}^\bullet &= (\ 0.342107,\ 0.328947,\ 0.328947);\\
|d_0^\bullet\rangle &\doteq {}^t[\ 0.363449,\ 0.000000,\ 0.931614],\\
|d_+^\bullet\rangle &\doteq {}^t[\ 0.658751,\ 0.707107, -0.256997],\\
|d_-^\bullet\rangle &\doteq {}^t[\ 0.658751, -0.707107, -0.256997];\\
\bar{P}_\mathrm{e}^\bullet &= 0.0941934,\\
\mathbf{u} &= (\ 0.333333,\ 0.333333,\ 0.333333);\\
\bar{P}_\mathrm{e}(\Pi^\bullet,\mathbf{u}) &= 0.0935778;\\
|d_0^\mathrm{bayes}(\mathbf{u})\rangle &\doteq {}^t[\ 0.355204,\ 0.000000,\ 0.934789],\\
|d_+^\mathrm{bayes}(\mathbf{u})\rangle &\doteq {}^t[\ 0.660996,\ 0.707107, -0.251167],\\
|d_-^\mathrm{bayes}(\mathbf{u})\rangle &\doteq {}^t[\ 0.660996, -0.707107, -0.251167];\\
\bar{P}_\mathrm{e}^\mathrm{bayes}(\mathbf{u}) &= 0.0935369.
\end{aligned}$$

The behaviors of $\bar{P}_\mathrm{e}^\mathrm{bayes}(\mathbf{p})$, $\bar{P}_\mathrm{e}(\Pi^\bullet,\mathbf{p})$, and $\bar{P}_\mathrm{e}(\Pi^\circ,\mathbf{p})$ at $\kappa = 0.5$ are shown in Fig.1 under the constraint $p_+ = p_-$. The minimum average probability of error $\bar{P}_\mathrm{e}^\mathrm{bayes}(\mathbf{p})$ draws a convex-upward curve, which is shown in the solid line. The dashed-dotted straight line stands for $\bar{P}_\mathrm{e}(\Pi^\bullet,\mathbf{p})$, the average probability of error of the square-root measurement at $\mathbf{p}$, which touches the convex curve

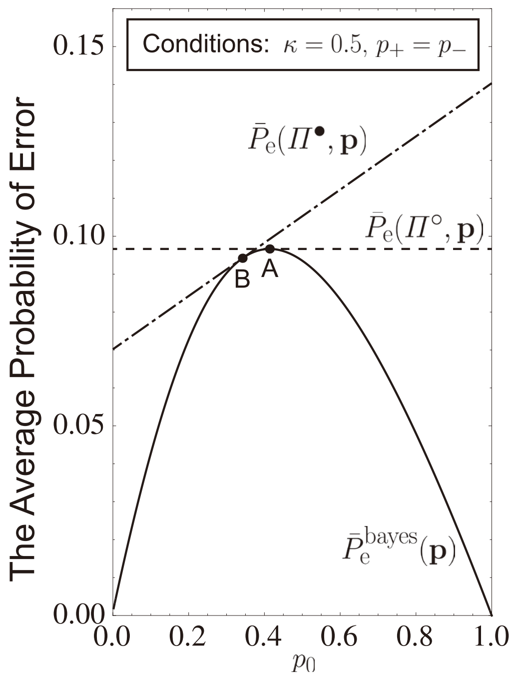

Fig. 1. $\bar{P}_{\mathrm{e}}^{\mathrm{bayes}}(\mathbf{p})$, $\bar{P}_{\mathrm{e}}(\Pi^\bullet, \mathbf{p})$, and $\bar{P}_{\mathrm{e}}(\Pi^\circ, \mathbf{p})$ for $\kappa = 0.5$, under the constraint $p_+ = p_-$. (Points C and D are omitted)

at the point $\mathsf{B}(0.342107, 0.0941934)$. The dashed line stands for the average probability of error of the minimax receiver, $\bar{P}_{\mathrm{e}}(\Pi^\circ, \mathbf{p})$. As expected from Lemma 3 of [4], this line remains constant and touches the convex curve at the point $\mathsf{A}(0.413815, 0.0966410)$. The point corresponding to $\bar{P}_{\mathrm{e}}(\Pi^\bullet, \mathbf{u})$ is $\mathsf{C}(0.333333, 0.0935778)$ on the dashed-dotted straight line $\bar{P}_{\mathrm{e}}(\Pi^\bullet, \mathbf{p})$, and the point corresponding to $\bar{P}_{\mathrm{e}}^{\mathrm{bayes}}(\mathbf{u})$ is $\mathsf{D}(0.333333, 0.0935369)$ on the solid line $\bar{P}_{\mathrm{e}}^{\mathrm{bayes}}(\mathbf{p})$. The points A, B and D are put on the convex curve of $\bar{P}_{\mathrm{e}}^{\mathrm{bayes}}(\mathbf{p})$, while the point C is not on the convex curve.

Since the point A is put on the top of the convex curve $\bar{P}_{\mathrm{e}}^{\mathrm{bayes}}(\mathbf{p})$, we observe $\bar{P}_{\mathrm{e}}^\circ > \bar{P}_{\mathrm{e}}^\bullet$ and $\bar{P}_{\mathrm{e}}^\circ > \bar{P}_{\mathrm{e}}^{\mathrm{bayes}}(\mathbf{u})$. Recall that the dashed-dotted straight line $\bar{P}_{\mathrm{e}}(\Pi^\bullet, \mathbf{p})$ is a tangent line to the convex curve $\bar{P}_{\mathrm{e}}^{\mathrm{bayes}}(\mathbf{p})$. This yields $\bar{P}_{\mathrm{e}}(\Pi^\bullet, \mathbf{u}) > \bar{P}_{\mathrm{e}}^{\mathrm{bayes}}(\mathbf{u})$. The relation $\bar{P}_{\mathrm{e}}^\bullet > \bar{P}_{\mathrm{e}}(\Pi^\bullet, \mathbf{u})$ comes from the fact that the slope of the tangent line $\bar{P}_{\mathrm{e}}(\Pi^\bullet, \mathbf{p})$ is positive and $1/3 < p_0^\bullet$.

V. Conclusions

The square-root measurement for the ternary coherent state signal $\{|0\rangle, |\alpha\rangle, |-\alpha\rangle\}$ was considered. For this signal, the closed-form expression of the square-root measurement $\Pi^\bullet$ is derived. Further, the optimal distribution $\mathbf{p}^\bullet$ of the signal that makes the square-root measurement Bayes-optimal is also derived. Through the numerical analysis, the minimax value $\bar{P}_{\mathrm{e}}^\circ$ was compared with the minimum average probability $\bar{P}_{\mathrm{e}}^\bullet$ of error at $\mathbf{p}^\bullet$, the average probability $\bar{P}_{\mathrm{e}}(\Pi^\bullet, \mathbf{u})$ of error obtained from the square-root measurement $\Pi^\bullet$ and the uniform distribution $\mathbf{u}$, and the minimum average probability $\bar{P}_{\mathrm{e}}^{\mathrm{bayes}}(\mathbf{u})$ of error at $\mathbf{u}$. From the comparison, we obtained the relation $\bar{P}_{\mathrm{e}}^\circ > \bar{P}_{\mathrm{e}}^\bullet > \bar{P}_{\mathrm{e}}(\Pi^\bullet, \mathbf{u}) \gtrsim \bar{P}_{\mathrm{e}}^{\mathrm{bayes}}(\mathbf{u})$ for $0 < \kappa < 1$ for the ternary coherent state signal. Thus it was numerically demonstrated that $\bar{P}_{\mathrm{e}}^\bullet$ provides a good lower bound of the minimax value $\bar{P}_{\mathrm{e}}^\circ$ for the ternary coherent state signal. More general discussion will be given in the near future.

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