Cut-off Rate for ASK signal states

Masaki Sohma

Quantum ICT Research Institute, Tamagawa University
6-1-1 Tamagawa-gakuen, Machida, Tokyo 194-8610, Japan

Cut-off Rate for ASK signal states

Musaki Sohma
Quantum Information Science Research Center, Quantum ICT Research Institute, Tamagawa University
6-1-1 Tamagawa-gakuen, Machida, Tokyo 194-8610, Japan
E-mail: sohma@eng.tamagawa.ac.jp

Abstract—We compute a cut-off rate $R_M$ for $M$-ary ASK signal states and deal with the discretization problem where we consider whether $R_M$ achieves a continuous cut-off rate.

I. INTRODUCTION

This paper discusses a cut-off rate of quantum Gaussian channels, where classical information is conveyed by quantum Gaussian states and a positive operator valued measure is used in the decoding procedure. In particular we mainly deal with ASK signal states.

The quantum cut-off rate is a quantity describing behavior of error probability exponents at medium rates[1]. The Holevo capacity gives the upper limit on the information rate for reliable classical-quantum communication and the quantum cut-off rate is considered to be the practical upper bound on the information rate of a classical-quantum channel[2]. Ban et.al developed a method of computing quantum cut-off rate and applied it for a group-covariant $M$-ary quantum state channel[2].

Our purpose is to reveal properties of ASK signal states in terms of quantum cut-off rate. Unfortunately, for ASK signal states, we cannot always apply the Ban’s method of computing a quantum cut-off rate. So we compute it by exploiting a numerical method in Sec. II. In Sec. III we obtain a continuous quantum cut-off rate in the case of one dimensional distributed signals and compare it with the result for the discrete case.

II. QUANTUM CUT-OFF RATE FOR ASK SIGNAL STATES

We firstly remind the quantum cut-off rate for classical-quantum communication channels with $M$ pure signal states $\{ |\psi_1 \rangle, \ldots, |\psi_M \rangle \}$. It is given by

$$ R_M = \max_{\pi} \tilde{\mu}(\pi, 1) $$

where the function $\tilde{\mu}(\pi, s)$ is a Gallager function given as

$$ \tilde{\mu}(\pi, s) = -s \ln \sum_{j=1}^{M} \sum_{k=1}^{M} \pi_j \pi_k |\langle \psi_j | \psi_k \rangle|^2/s. $$

Ban found that the quantum cut-off rate can be computed as

$$ R_M = \ln \left( \sum_{j=1}^{M} \sum_{k=1}^{M} (G_2^{-1})_{jk} \right), $$

if

$$ \tilde{\pi}_j = \frac{\sum_{m=1}^{M} (G_2^{-1})_{jk}}{\sum_{k=1}^{N} (G_2^{-1})_{jk}}, $$

is non-negative for all $j = 1, 2, \ldots, M$, where $G_2^{-1}$ is the inverse of the matrix $(G_2)_{jk} = |\langle \psi_j | \psi_k \rangle|^2$ [2]. Note that $\{ \tilde{\pi}_j \}$ gives the optimum input probability when the above condition is satisfied.

Let us compute the quantum cut-off rate for ASK signal states, which consists of $M$ signal states $\{-\alpha, \ldots, |\alpha|\}$. Here we assume $\alpha$ is a real number for simplicity. Unfortunately we cannot always employ Ban’s formula (3) because $\tilde{\pi}_j$ may not be positive when distance between signals is short. Then we must rely on numerical computation.

In Fig. 1 circles are computed by Ban’s formula. The graphs indicate that large number of signals is needless. Unlike the case of PSK signal states the average energy

$$ N = \sum_{j=1}^{M} \pi_j |\alpha_j|^2 $$

with $\{ |\alpha_1 \rangle, \ldots, |\alpha_M \rangle \} = \{-\alpha, \ldots, |\alpha|\}$, changes as a priori probability distribution $\pi$ does. We are interested in knowing how the average energy, $N_M$, for the optimum a priori distribution changes according to the number of signals, $M$. Fig. 2 shows the graph of $N_M/N_2$, with respect to number of signals $M$ for $\alpha = 2, 5, 10$. Here we use a normalization $N_M/N_2$ instead of $N_M$, because we are interested in whether we need a larger energy when $M$ takes a larger value.

III. DISCRETIZATION

We remind the quantum cut-off rate for a continuous classical-quantum channel with pure signal states $\{ |\psi_m \rangle ; m \in \mathcal{M} \}$ where $\mathcal{M}$ is a Borel subset in a finite dimensional Euclidean space. In [6] it is given as

$$ R_C = \max_{0 \leq \mu \leq P, \pi \in \mathcal{P}_1} \tilde{\mu}(\pi, 1, p), $$

where $\mathcal{P}_1$ is the set of probability distribution $\pi$ satisfying $\int f(m)\pi(dm) \leq E$ for a fixed nonnegative Borel function $f$ on $\mathcal{M}$ and

$$ \tilde{\mu}(\pi, s, p) = -s \ln \int e^p [f(m) + f(\alpha) - 2E] |\langle \psi_m | \psi_n \rangle|^2/s \pi(dm) \pi(dn). $$


This is the quantum cut-off rate that we can achieve if we are allowed to use codes \(|\psi_{m_1}\otimes \cdots \otimes |\psi_{m_K}\rangle\) satisfying energy constraint

\[
f(m_1) + \cdots + f(m_K) \leq KE. \tag{8}
\]

On the other hand in the case of quantum cut-off rate \(R_M\) any letter states \(|\psi_{m_j}\rangle\) in a codeword are chosen from the fixed finite set \(\{|\psi_1\rangle, \ldots, |\psi_M\rangle\}\) and energy constraint is not considered. Putting \(p = 0\) and considering a discrete probability distribution as \(\pi\) in Eq. (6), we obtain the following relation

\[
R_M \leq R_C, \tag{9}
\]

where \(\{|\psi_1\rangle, \ldots, |\psi_M\rangle\} \subset \{|\psi_m\rangle: m \in M\}\) and energy constraint \(E\) is fixed to the value of average energy with optimum probability distribution in Eq. (1).

In the following we devote ourselves to the case of coherent signal states. Then we consider

\[
f(\alpha) = \hbar |\alpha|^2 \tag{10}
\]
as a signal energy for coherent state \(|\alpha\rangle\) and put \(E = \hbar N_{tr}\).

Let us compute the Gallager function assuming a priori probability distribution is Gaussian

\[
\pi(d^2\alpha) = \frac{1}{\pi N_{tr}} \exp\left(-\frac{|\alpha|^2}{N_{tr}}\right) d^2\alpha, \tag{11}
\]

where \(\alpha\) is a complex valued random variable. Then we have [6]

\[
\bar{\mu}(\pi, s, p) = s \left[2pE + \log\left(1 + p^2E^2 - 2pE + \frac{E(1-pE)}{\hbar s}\right)\right],
\]

with \(E = \hbar N_{tr}\), and we can solve the optimization (6) and obtain

\[
R_C = 2N_{tr} + 2 - \vartheta(2N_{tr}) + \ln \vartheta(2N_{tr}), \tag{13}
\]

with \(\vartheta(t) = (1 + \sqrt{t^2 + 1})/2\). It is more suitable to consider the case where a priori probability \(\pi\) is distributed one-dimensionally:

\[
\pi(dx) = \frac{1}{\sqrt{2\pi N_{tr}}} \exp\left(-\frac{|x|^2}{2N_{tr}}\right) dx. \tag{14}
\]

Then we compute the cut-off rate \(R_x\) as

\[
R_x = 2N_{tr} + 1 - \vartheta(4N_{tr}) + \frac{\ln \vartheta(4N_{tr})}{2}. \tag{15}
\]

Here we have the relation \(R_M \leq R_x \leq R_C\) and \(R_M (M = 2, 3, \ldots)\) increases monotonously with respect to \(M\). Fig. 3 shows the graphs of \(R_2 / R_x\) and \(R_{30} / R_x\) with respect to average signal energy \(N_{tr}\). Note that we have \(N_M \leq N_{30}\) for \(M < 30\) and \(N_M = N_{30}\) for \(M > 30\) in our case, where \(N_{tr}\) is small (\(N_{tr} < 0.8\)). These graphs show the followings.

1) When the average energy \(N_{tr}\) is small (\(N_{tr} < 0.1\)), the binary cut-off rate \(R_2\) has almost the same value as the continuous cut-off rate \(R_x\).

2) When \(N_{tr} < 0.6\), it is not necessary to use more than two signal states, i.e. \(N_M = N_{30}, M = 3, 4, \ldots\)

3) When \(N_{tr}\) has the large value, the value of \(R_M\) does not approach to that of \(R_x\) even if we take any large values of \(M\).

By further computation we can find graphs of \(R_M / R_x\). \(M \geq 3\) coincide each other when \(N_{tr}\) is small (e.g. \(N_{tr} \leq \)).
Fig. 3. Dependence of ratios $R_2/R_x, R_30/R_x$ on average signal energy, $N_{tr}$.

0.8). So the dot line in Fig. 3 gives an upper bound. On the other hand for a larger value of $N_{tr}$ we need a larger value of $M$ in order to achieve the upper bound.

IV. CONCLUSION

We have computed the cut-off rate $R_M$ for $M$-ary ASK signal states $\{|-\alpha\rangle, \ldots, |\alpha\rangle\}$ and compared it with the continuous cut-off rate $R_x$. The value of binary cut-off rate $R_2$ is equal to that of $R_x$ approximately when $N_{tr} << 1$ while $R_M$ does not achieve the continuous cut-off rate even if we take any large values of $M$. This means that our strategy based on ASK signal states is not suitable to achieve the continuous cut-off rate for a large value of $N_{tr}$.

REFERENCES