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***Abstract*—We report on verification of the relation between Holevo's commutation operator and the modular operator.**

I. Introduction

In the study of noncommutative statistics, Holevo introduced the space of square-integrable operators and the associated superoperators called the commutation operators [1]. These are very useful mathematical tools for the solution of noncommutative statistical problems. On the other hand the modular operator is well known in the operator theory; it appears in the main results of the Tomita-Takesaki theory. Holevo pointed out that we can show the relation between the commutation operator and the modular operator by a simple computation [1]. This paper reviews results about theses operators in the simplest case, and gives a detailed computation to verify the relation between them.

We mainly deal with the von Neumann algebra $\mathfrak{N} = M_n(\mathbb{C})$, which is the ensemble of $n \times n$ complex matrices and can be considered as the algebra $\mathfrak{B}(\mathcal{H})$ of bounded linear operators on the Hilbert space $\mathcal{H} = \mathbb{C}^n$ with the inner product $(x, y) = \bar{x}^T y$. Let ρ be a non-degenerated density operator and ω be the corresponding normal state given by $\omega(A) = \mathrm{Tr}\rho A, A \in \mathfrak{N}$. We regard $\mathfrak{N}$ as a Hilbert space with the inner product

$$\langle A, B\rangle = \frac{1}{2}\omega(BA^* + A^*B), \tag{1}$$

and denote it by $\mathfrak{H}$. In the quantum theory we consider additional bilinear form on $\mathfrak{H}$

$$[A, B] = i\omega(A^*B - BA^*). \tag{2}$$

and obtain fundamental inequalities

$$\langle X, X\rangle \geq \pm\frac{i}{2}[X, X],$$

which yield the uncertainty relation of the most general form [1]. We define a commutation operator $\mathfrak{D}$ so that it satisfies

$$[A, X] = \langle A, \mathfrak{D}X\rangle. \tag{3}$$

The operator $\mathfrak{D}$, firstly introduced by Holevo [1], plays an important role in the non-commutative statistical theory. From Eq. (2) it holds that

$$1 \pm \frac{i}{2}\mathfrak{D} \geq 0. \tag{4}$$

On the other hand we can also regard $\mathfrak{N}$ as a Hilbert-Schmidt space with another inner product

$$\langle A, B\rangle_2 = \mathrm{Tr}(A^*B),$$

by virtue of its finite-dimensionality and denote it by $\mathfrak{H}_2$. Let us consider a $*$-representation on $\mathfrak{H}_2$

$$\ell : \mathfrak{N} \to \mathfrak{B}(\mathfrak{H}_2), \tag{5}$$

where $\mathfrak{B}(\mathfrak{H}_2)$ is the ensemble of bounded operators on $\mathfrak{H}_2$ and $\ell(A)B = AB$. Then the state ω can be written by the inner product $\langle\cdot,\cdot\rangle_2$ as

$$\omega(A) = \mathrm{Tr}(\rho^{1/2}A\rho^{1/2}) = \langle\rho^{1/2}, \ell(A)\rho^{1/2}\rangle_2.$$

Remark: In the present case we have $\mathfrak{H}_2 = \mathfrak{H}$, but when we consider an infinite dimensional Hilbert space $\mathcal{H}$ the equality does not hold, i.e. $\mathfrak{H}_2 \subset \mathfrak{B}(\mathcal{H}) \subset \mathfrak{H}$. This may make it difficult to extend the discussion in Sec. 4 to the infinite dimensional case.

As stated in Sec. 3, we can define the modular operator Δ for the von Neumann algebra $\mathfrak{M} = \ell(\mathfrak{N})$ and its cyclic separating vector $\rho^{1/2} \in \mathfrak{H}_2$. In this paper we derive a simple relation between such derived modular operator and the commutation operator:

$$\Delta = \left(1 + \frac{i}{2}\mathfrak{D}\right)\left(1 - \frac{i}{2}\mathfrak{D}\right)^{-1},$$

which is originally shown in [1].

II. Representation on $\mathcal{H} \otimes \mathcal{H}$

We identify the Hilbert-Schmidt space $\mathfrak{H}_2 (= \mathfrak{N} = M_n(\mathbb{C}))$ on $\mathcal{H} = \mathbb{C}^n$ with $\mathcal{H} \otimes \mathcal{H}$ by a unitary operator v satisfying

$$v(e_j e_k^T) = e_j \otimes e_k,$$

where $e_j = (\delta_{j1}, \delta_{j2}, ..., \delta_{jn})^T$ with the Kronecker delta δ_{jl}. Here, for $\psi = \sum_j \lambda_j e_j$ and $\phi = \sum_k \mu_k e_k$ we have

$$\begin{aligned} v(\psi\phi^*) &= \sum_{j,k}\lambda_j\bar{\mu}_k v(e_j e_k^T) \\ &= \sum_{j,k}\lambda_j\bar{\mu}_k e_j \otimes e_k = \psi \otimes \overline{\phi}. \end{aligned} \tag{6}$$

In the $*$-representation (5), $\ell(A)$ is given by $\tilde{\ell}(A) = A \otimes I_n$ on $\mathcal{H} \otimes \mathcal{H}$. In fact

$$\begin{aligned} v(\ell(A)e_j e_k^T) &= v(Ae_j e_k^T) = Ae_j \otimes e_k \\ &= (A \otimes I_n)e_k \otimes e_j = \tilde{\ell}(A)v(e_j e_k^T), \end{aligned} \tag{7}$$

and hence $\ell(A) = v^*\tilde{\ell}(A)v$.
On the other hand,

$$r(A) : \mathfrak{N} \in X \to XA \ni \mathfrak{N}$$

is represented by $\tilde{r}(A) = I_n \otimes A^T$ on $\mathcal{H} \otimes \mathcal{H}$. In fact

$$\begin{aligned} v(r(A)e_j e_k^T) &= v(e_j e_k^T A) = v(e_j (A^* e_k)^*) \\ &= e_j \otimes \overline{A^* e_k} = e_j \otimes A^T e_k \\ &= (I_n \otimes A^T) e_j \otimes e_k = \tilde{r}(A) v(e_j e_k^T), \end{aligned} \tag{8}$$

and hence $r(A) = v^*\tilde{r}(A)v$. Remark that $\tilde{\ell}(\mathfrak{N}) = \mathfrak{N} \otimes I_n$ and $\tilde{r}(\mathfrak{N}) = I_n \otimes \mathfrak{N}$ are von Neumann algebras in

$$\mathfrak{B}(\mathcal{H} \otimes \mathcal{H}) = \mathfrak{B}(\mathcal{H}) \otimes \mathfrak{B}(\mathcal{H}) = \mathfrak{N} \otimes \mathfrak{N},$$

and we have

$$\tilde{\ell}(\mathfrak{N})' = \tilde{r}(\mathfrak{N}), \quad \tilde{r}(\mathfrak{N})' = \tilde{\ell}(\mathfrak{N}). \tag{9}$$

III. Modular Operator

We give a proof of the main results of Tomita-Takesaki theory in the case of $\mathfrak{M} = \ell(\mathfrak{N}) \subset \mathfrak{B}(\mathfrak{H}_2)$ with $\mathfrak{N} = M_n(\mathbb{C})$, where all difficulties in the theory vanish. From Eq. (9), we have

$$\begin{aligned} \mathfrak{M}' &= v^*\tilde{\ell}(\mathfrak{N})'v = v^*\tilde{r}(\mathfrak{N})v = r(\mathfrak{N}) \\ \mathfrak{M}'' &= v^*\tilde{r}(\mathfrak{N})'v = v^*\tilde{\ell}(\mathfrak{N})v = \ell(\mathfrak{N}) = \mathfrak{M}. \end{aligned} \tag{10}$$

We introduce a cyclic separating vector $\rho^{1/2}$, satisfying

$$\mathfrak{H}_2 = \mathfrak{M}\rho^{1/2} = \mathfrak{M}'\rho^{1/2},$$

and consider anti-linear operators on $\mathfrak{H}_2$

$$S : \ell(A)\rho^{1/2} \to \ell(A)^*\rho^{1/2}, \tag{11}$$

$$F : r(A)\rho^{1/2} \to r(A)^*\rho^{1/2}. \tag{12}$$

Here

$$\ell(A)^* = v^*(A \otimes I_n)^* v = v^*\tilde{\ell}(A^*)v = \ell(A^*),$$

and

$$r(A)^* = v^*(I_n \otimes A^T)^* v = v^*\tilde{r}(A^*)v = r(A^*).$$

The linear operator $\Delta = FS$ on $\mathfrak{H}_2$ is known as a modular operator. For the operator S, we have

$$S(X) = \rho^{-1/2} X^* \rho^{1/2}.$$

In fact, putting

$$\begin{aligned} X &= \ell(A)\rho^{1/2} = A\rho^{1/2}, \\ Y &= \ell(A)^*\rho^{1/2} = \ell(A^*)\rho^{1/2} = A^*\rho^{1/2}, \end{aligned} \tag{13}$$

we have

$$Y = (X\rho^{-1/2})^*\rho^{1/2} = \rho^{-1/2}X^*\rho^{1/2}.$$

On the other hand, for the operator F, we have

$$F(X) = \rho^{1/2}X^*\rho^{-1/2}.$$

In fact, putting

$$\begin{aligned} X &= r(A)\rho^{1/2} = \rho^{1/2}A, \\ Y &= r(A)^*\rho^{1/2} = r(A^*)\rho^{1/2} = \rho^{1/2}A^*, \end{aligned} \tag{14}$$

we have

$$Y = \rho^{1/2}(\rho^{-1/2}X)^* = \rho^{1/2}X^*\rho^{-1/2}.$$

Thus we have

$$\begin{aligned} \Delta(X) &= FS(X) = F(\rho^{-1/2}X^*\rho^{1/2}) \\ &= \rho^{1/2}(\rho^{-1/2}X^*\rho^{1/2})^*\rho^{-1/2} \\ &= \rho X \rho^{-1}, \end{aligned} \tag{15}$$

we have that is,

$$\Delta = v^*(\rho \otimes (\rho^{-1})^T)v. \tag{16}$$

It follows that $\Delta^* = v^*(\rho \otimes (\rho^{-1})^T)^* v = \Delta$. Since $\Delta^{-1/2} = v^*(\rho^{-1/2} \otimes (\rho^{1/2})^T)v$,

$$\Delta^{-1/2}(X) = \rho^{-1/2}X\rho^{1/2}$$

and hence

$$S(X) = \Delta^{-1/2}(X^*) = \Delta^{-1/2}J(X),$$

where J is an anti-linear operator defined by $J(X) = X^*$. In a similar way we have

$$F(X) = \Delta^{1/2}J(X).$$

Since

$$\begin{aligned} \Delta^{-it}\ell(A)\Delta^{it}(X) &= \Delta^{-it}(A\rho^{it}X\rho^{-it}) = \rho^{-it}A\rho^{it}X\rho^{-it}\rho^{it} \\ &= \rho^{-it}A\rho^{it}X = \ell(\rho^{-it}A\rho^{it})(X), \end{aligned} \tag{17}$$

we have

$$\Delta^{-it}\ell(A)\Delta^{it} = \ell(\rho^{-it}A\rho^{it}).$$

On the other hand, we have

$$J\ell(A)J(X) = J(AX^*) = (AX^*)^* = XA^* = r(A^*)(X)$$

Thus we obtain the main results of Tomita-Takesaki theory in our case:

$$\Delta^{-it}\mathfrak{M}\Delta^{it} = \mathfrak{M}, \tag{18}$$

$$J\mathfrak{M}J = \mathfrak{M}'. \tag{19}$$

Remark that the above discussion can be easily extend to the case where $\mathfrak{N} = M_{m_1}(\mathbb{C}) \oplus \cdots \oplus M_{m_n}(\mathbb{C})$. The proof of Tomita-Takesaki theory for a finite dimensional von Neumann algebra is given in the Appendix.

IV. Relation between Holevo's commutation operator and modular operator

Let us see how the commutation operator defined by (3) is described on $\mathcal{H} \otimes \mathcal{H}$. Since it holds for $A, X, Y = \mathfrak{D}X \in \mathfrak{H}(= \mathfrak{N} = \mathfrak{H}_2)$ that

$$\begin{aligned} [A, X] &= i\omega(A^*X - XA^*) = i\mathrm{Tr}\rho(A^*X - XA^*) \\ &= \mathrm{Tr}A^* i(X\rho - \rho X) = \langle A, i(X\rho - \rho X)\rangle_2 \\ \langle A, Y\rangle &= \omega((YA^* + A^*Y)/2) = \mathrm{Tr}\rho((YA^* + A^*Y)/2) \\ &= \mathrm{Tr}A^*(\rho Y + Y\rho)/2 = \langle A, \rho Y + Y\rho\rangle_2, \end{aligned} \tag{20}$$

we have

$$(\rho Y + Y\rho)/2 = i(X\rho - \rho X),$$

which can be represented on $\mathcal{H}\otimes\mathcal{H}$ as

$$(\rho\otimes I_n + I_n\otimes\rho^T)v(Y) = 2i(I_n\otimes\rho^T - \rho\otimes I_n)v(X).$$

Thus

$$v(\mathfrak{D}X) = v(Y) = 2i(\rho\otimes I_n + I_n\otimes\rho^T)^{-1}(I_n\otimes\rho^T - \rho\otimes I_n)v(X),$$

that is,

$$\mathfrak{D} = v^*[2i(\rho\otimes I_n + I_n\otimes\rho^T)^{-1}(I_n\otimes\rho^T - \rho\otimes I_n)]v.$$

Moreover

$$1+\frac{i}{2}\mathfrak{D} = v^*[2(\rho\otimes I_n + I_n\otimes\rho)^{-1}\rho\otimes I_n]v, \tag{21}$$

$$1-\frac{i}{2}\mathfrak{D} = v^*[2(\rho\otimes I_n + I_n\otimes\rho)^{-1}I_n\otimes\rho^T]v, \tag{22}$$

$$1+\frac{1}{4}\mathfrak{D}^2 = v^*[4(\rho\otimes I_n + I_n\otimes\rho)^{-2}\rho\otimes\rho^T]v. \tag{23}$$

From our assumption stated in Sec. 1, $\rho^{1/2}$ is a non-degenerated operator and hence we can use it as a cyclic separating vector in Sec. 3. Thus, from Eqs. (21), (22) and (16) we conclude

$$\Delta = \left(1+\frac{i}{2}\mathfrak{D}\right)\left(1-\frac{i}{2}\mathfrak{D}\right)^{-1},$$

and

$$\frac{i}{2}\mathfrak{D} = (\Delta-1)(\Delta+1)^{-1}.$$

V. Appendix

A Proof of general Tomita-Takesaki theory was given by [3]. Afterward Longo shown that its proof can be simplified for approximately finite von Neumann algebra [2]. In this section we prove the main results of Tomita-Takesaki theorem for a finite dimensional von Neumann algebras $\tilde{\mathfrak{N}}$ on a Hilbert space $\mathcal{K}$ according to [2] for readers' convenience. We assume there exists a cyclic separating vector $\xi\in\mathcal{K}$; $\mathcal{K}=\tilde{\mathfrak{N}}\xi=\tilde{\mathfrak{N}}'\xi$. Then we define operators $\tilde{S},\tilde{F},\tilde{\Delta}$ and $\tilde{J}$ on $\mathcal{K}$ as

$$\begin{aligned}&\tilde{S}: X\xi\to X^*\xi, X\in\tilde{\mathfrak{N}}\\ &\tilde{F}: Y\xi\to Y^*\xi, Y\in\tilde{\mathfrak{N}}'\\ &\tilde{\Delta}=\tilde{F}\tilde{S},\\ &\tilde{J}=\tilde{\Delta}^{1/2}\tilde{S}.\end{aligned} \tag{24}$$

The Wedderburn theorem states that a finite dimensional C^* algebra is $*$-isomorphic to a direct sum of simple matrix algebras. That is, there exists $*$-isomorphic function φ for von Neumann algebra $\tilde{\mathfrak{N}}$ such that

$$\varphi:\tilde{\mathfrak{N}}\simeq\mathfrak{N} := M_{m_1}(\mathbb{C})\oplus\cdots\oplus M_{m_n}(\mathbb{C}).$$

Let us consider the faithful state on $\tilde{\mathfrak{N}}$ as

$$\omega_\xi(\tilde{A}) = (\xi,\tilde{A}\xi)_{\mathcal{K}},\quad \tilde{A}\in\tilde{\mathfrak{N}},$$

where $(\cdot,\cdot)_{\mathcal{K}}$ is an inner product of the Hilbert space $\mathcal{K}$. Using this state we can define the state on $\mathfrak{N}$

$$\omega(A) = \omega_\xi(\varphi^{-1}(A)), A\in\mathfrak{N},$$

which is normal by virtue of finite-dimensionality and is faithful because ξ is separating, i.e. there exists a non-degenerated density operator ρ such that $\omega(A) = \mathrm{Tr}\rho A$. Applying the discussion in Sec. 3 to the von Neumann algebra $\mathfrak{M}=\ell(\mathfrak{N})$ and the cyclic separating vector $\rho^{1/2}$, we get the operators, S,F,J and Δ. In particular the operators S and F are given by Eqs. (11) and (12). Here we have

$$\langle\ell(A)\rho^{1/2},\ell(B)\rho^{1/2}\rangle_2 = (\varphi^{-1}(A)\xi,\varphi^{-1}(B)\xi)_{\mathcal{K}},$$

which means

$$U:\mathfrak{H}_2\ni\ell(A)\rho^{1/2}\to\varphi^{-1}(A)\xi\in\mathcal{K},\quad A\in\mathfrak{N}$$

gives a unitary operator from $\mathfrak{H}_2$ to $\mathcal{K}$. Using this unitary operator we obtain the following relations

$$\begin{aligned}\tilde{S}&=USU^*,\\ \tilde{F}&=UFU^*,\end{aligned} \tag{25}$$

and hence

$$\tilde{\Delta}=U\Delta U^*, \tag{26}$$

$$\tilde{J}=UJU^*. \tag{27}$$

Moreover

$$\tilde{\mathfrak{N}}=U\mathfrak{M}U^*, \tag{28}$$

since $\mathfrak{M}=\ell(\mathfrak{N})$ and $\varphi^{-1}(A)=U\ell(A)U^*$ for $A\in\mathfrak{N}$. From Eqs. (18),(19),(26),(27) and (28), we conclude the main result of Tomita-Takesaki theory,

$$\begin{aligned}\tilde{\Delta}^{-it}\tilde{\mathfrak{N}}\tilde{\Delta}^{it}&=\tilde{\mathfrak{N}}\\ \tilde{J}\tilde{\mathfrak{N}}\tilde{J}&=\tilde{\mathfrak{N}}'.\end{aligned} \tag{29}$$

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References

[1] A. S. Holevo,Commutation superoperator of a state and its applications to the noncommutative statistics, Reports on Mathematical Physics,251-271,1977

[2] Roberto Longo, A simple proof of the existence of modular automorphisms in approximately finite-dimensional von Neumann algebras, 199-205,1978

[3] M. Takesaki, Tomita's Theory of Modular Hilbert Algebras and its Application, Springer, 1970