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Basis of quantum noise analysis for quantum computers

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***Abstract*—It is expected that quantum computers provide very attractive capabilities for real applications, and our common challenge is to realize them as real-world technologies. In the past, the prediction of functions in an ideal environment has been prioritized, and issues at the implementation stage have not been much of a problem. Thus it is necessary to study in detail whether an actual quantum computer with the capability of cryptanalysis is feasible as an extension of the present implementation technology. The first task is to fully model and characterize the natures of quantum noise in large-scale quantum CPUs. In this paper, we presents a classification method of quantum noise from an information-theoretic perspective, and introduces specific physical phenomena of nonlinear and nonlocal errors, which are unique quantum noise phenomena. Finally, as our main results, we give a method of information-theoretic modeling of these phenomena, showing several examples.**

I. Introduction

Quantum computers are theoretically predicted to have significantly higher computing power than conventional computers [1], and the standardization of post-quantum public key cryptography [2] has been started in conjunction with the development of quantum computer. Recent discussions on the demonstration of quantum transcendence by Google and IBM have sparked public interest in the real performance of quantum computers. A quantum computer must realize a combinatorial circuit of large number of quantum gates and quantum memories. However, a mathematical proof called the threshold theorem [3] shows a prediction law comparable to Moore's law for classical computers. According to it, it will take a similar evolutionary process to the classical computer from the current small-scale quantum computer. However, the situation is not so simple. That is, many researchers are concerned about the current theoretical analysis of the unavoidable quantum noise in the implementation of large scale quantum CPUs for practical quantum computers [4,5,6,7,8]. In particular, Nobel laureates in physics, such as S. Haroche, have questioned the current way of thinking on the implimentation of quantum CPU. The concern is that existing theory is based on optimistic reasoning based on a lack of accurate understanding of quantum noise in quantum circuits of quantum computer architecture. Thus, system architecture researchers and others have specifically conducted research to confirm their concerns. We are concerned with more general theory on the quantum noise analysis. In this paper, we firstly present the most general classification of errors by quantum noise based on the results of considering the standard mathematical models of quantum noise analysis. It is defined by the information theoretic point of view. From such a formulation, one can see that the probability of error of a quantum bit depends on the number of qubits in a quantum CPU. This is a new type of noise come from the quantum nature. It will be called "Nonlinear error". Then, real physical phenomena that cause errors in the such a case are introduced. Based on the above theory, we propose an information-theoretic modeling of such quantum-specific noise effects and give several examples. Consequently, if the nonlinear error occurs in quantum computer, the threshold theorem cannot hold its claim. So we have to point out the fact that the implementation of quantum computer with error correcting function is impossible. Since such studies will influence the development of post-quantum public key cryptography and physical cryptography, we hope that the contents of this paper will provide an appropriate information to researchers in related fields.

Finally we emphasize that the quantum noise for quantum computers is an essential issue in order to put quantum computers into practical use at a truly useful level. To cope with these difficulties, it is necessary to bring together the wisdom of not only physicists but also information theory / code theory researchers and computer scientists who are not familiar with quantum mechanics. Thus it is of great significance to build a quantum computer noise model that can be understood without the detailed physical phenomena, which this paper is trying.

II. Information theoretic view of quantum error

A. Phenomena of quantum noise in quantum computer

The quantum computer operates as a time evolution of quantum states in Hibert space. However, we need a method to mathematically model these quantum states, including the possibility of their involuntary temporal evolution. In this section, we classify the quantum noise

effects on quantum states and discuss the corresponding physical phenomena in order to make it easier for information theorists to participate. This has been clarified in 2019 by the present author [9,10]. Let us surmarize the concept. First of all, the quantum noise we are discussing here refers to the phenomenon of decoherence to quantum states. The following is a list of classifications as causes of quantum noise, specific noise phenomenal classifications, and information-theoretic error forms, respectively.
First, the causal classification is as follows.
(1) Interaction with environment, (2) Interaction with other qubits, (3) Imperfect gate, (4) Leakage.
The physical phenomena, on the other hand, are as follows.
(1) Stochastic Pauli Noise: Corresponding to bit or phase errors of a single qubit.
(2) Coherent Noise: No decoherence to a quantum state occurs, but it becomes an unintended quantum state.
(3) Amplitude Damping: a specific example of decoherence, especially derived from energy loss.
(4) Local correlated noise (Markov, non-Markov):
It is a statistically independent extension of Pauli noise, in which several qubits around the errored qubits are correlated to produce the error.
(5) Non-local correlated noise (Markov, non-Markov):
It also gives an error for every qubit in the system with correlation.
Next, their effects can be categorized as follows.
(1) Individual error
(2) Coherent error : error that preserves unitary nature
(3) Burst error : many successive bits are in error at the same time
(4) Synchronization avalanche error : destructive error due to the avalanche phenomenon

Why it is necessary to consider the above issues in the case of quantum computers is due to the following reasons. In the classical system, the semiconductor elements that make up a bit can be considered to be independent of each other. Next, the noise is additive, and errors in the execution of logical calculations are sufficiently practical to be analyzed only by the stochastic properties of the noise itself. As a result, almost all errors can be considered to be each bit independent or, if correlated, very local. On the other hand, in a quantum computer, most qubits are coupled by quantum correlations such as entanglements, so it can be special that only some qubits make errors independently.

B. Calssification of quantum errors in information theory

In this section, we present an information-theoretic classification of errors that occur in quantum computers, which is the subject of this paper. In information theory, the probability of the occurrence of an error is an important parameter. Therefore, any physical phenomenon that produces an error is acceptable, and as a result we classify information errors according to what characteristics they have.

$(\mathbf{1})$ Linear individual error
Assume that N qubits are prepared. Errors shall occur separately and independently of the quantum state of each of its qubits. Let the basic error probability for each qubit be $p(error)$, and when this probability does not depend on the number of qubits, we call it linear, and the single and T error probability in N qubits are

$$p(error) \equiv \eta_j = \eta^* \quad \forall j \in N \tag{1}$$
$$P_e(T) \propto {\eta^*}^T (1-\eta^*)^{N-T} \tag{2}$$

$(\mathbf{2})$ Linear local correlation and nonlocal correlation error
Assume that N qubits are prepared, and if an error occurs in the quantum state of one qubit with probability η_j, the error or decay of T qubits is induced by correlation of neighboring (local) or arbitrary (non-local) qubits of the system. This is called a correlation error. Here, assuming that the error probability η_j for each qubit does not depend on the number of qubits, the correlation error probability for a group of correlated qubits is

$$P_e(T) = \eta_j = \eta^* \tag{3}$$

$(\mathbf{3})$ Nonlinear individual error
We assume that N qubits are provided and that errors occur separately and independently in the quantum states of each qubit. However, if we assume that the probability of error for each qubit depends on the number of qubits, a single or T qubits error can be described as follows.

$$p(error) = \eta_j(N) = f(\eta^*, N) \sim \eta^* N^\alpha \tag{4}$$
$$P_e(T) \propto {\eta_j}^T(N)(1-\eta_j(N))^{N-T} \tag{5}$$

where α is a real number for specific approximation. This is called nonlinear error.

$(\mathbf{4})$ Nonlinear local and nonlocal correlation error
Suppose that N qubits are prepared, and if an error occurs in the quantum state of one qubit, the error occurs in the neighborhood or the whole system with correlation. And the probability of the error of a single qubit triggering it depends on the number of qubits, as in Equation (4), and it can be described by

$$p(error) = \eta_j(N) = f(\eta^*, N) \sim \eta^* N^\alpha \tag{6}$$

$(\mathbf{5})$ Avalanche error and accumulation error
If a correlation error occurs where the whole system loses its quantum nature, it is called an "avalanche collapse". We also define a propagation-accumulation error as the time when the first error propagates to the next step in an iterative gate operation or an iterative calculation and the error accumulates in the quantum circuit where the various errors mentioned above occur.

III. Basis of quantum noise analysis

A quantum computation mechanism has a structure in which all qubits in a quantum CPU are correlated and a huge pure quantum state consisting of all qubits is unitarily evolved according to a program using the correlation. In other words, the whole CPU is considered to be monolithic, so the interaction between the pure-state system and the environment including the vacuum field will inevitably cause the quantum states that carry information to become undesired quantum states or to be destroyed. Then, simple bit-flip and phase flip (Pauli-flip type) errors similar to classical systems are rather exceptional, and quantum-specific errors can be the main ones. Therefore, in order to predict the realization of a large-scale quantum computing mechanism, it is essential to elucidate the exact features of the noise itself by rigorous quantum noise analysis. The following is a starting point for this. First, let X be a physical quantity representing a quantum bit. Let the noise operator representing the noise for a quantum bit be $\mathcal{N}$. Here the interaction between qubits, which are information, and the interaction between qubits and noise is quite different from the classical system. The analysis of the characteristics of these interactions is called quantum noise analysis. The interaction is denoted by the interaction Hamiltonian H_{int}, which consists of X and $\mathcal{N}$. The Hamiltonian of the entire system is as follows.

$$H = H_X + H_{\mathcal{N}} + H_{int} \equiv H_0 + H_{int} \tag{7}$$

The quantum state representing the information evolves in time driven by the above Hamiltonian, but depending on the situation, the equation is either the Schrödinger equation on the extended Hilbert space, or the following Lindblad equation

$$\frac{\partial \rho}{\partial t} = \frac{-i}{\hbar}[H, \rho] + \sum_{i=1}^{N^2-1} \gamma_i (L_i \rho L_i^{\dagger} - \{L_i^{\dagger} L_i, \rho\}) \tag{8}$$

where L is a Lindblad-decoherence operator. Currently, this equation is the most frequently utilized. Assuming an actual general-purpose program, further generalization to include measurement systems is needed, not only models of decoherence systems as described in the previous section. As the generalization, it is necessary to introduce the following Belavkin equation [11].

$$d\psi = -(\frac{1}{2}\mathcal{L}^* \mathcal{L} + \frac{i}{\hbar} H)\psi dt + \mathcal{L}\psi dy \tag{9}$$

$$y(t) = \int_0^t dy(r), \tag{10}$$

We will discuss such a generalization in the next paper.

IV. Physical examples of quantum noise

This section shows physical eaxamples [4,5,6] in the error model categorized in section 2. These quantum noises have been not assumed in the conventional threshold theorem[3]. However, because physical phenomena are so complex, we exclude physical rigor and emphasize the logic that arrives at each error model. That is, the main objective of this section is to show that the new error models in Section 2 exist in reality, or rather, that they can be the main noise.

A. Hutter-Loss recurrence effect

Consider a model in which a group of qubits coupled by quantum correlations are interacted to the heat bath of a considered environment. There are many physical mechanisms out there, such as direct interactions between each qubit and the heat bath, and non-Markovian interactions mediated by the heat bath. First, however, we focus on the simplest of phenomena. That is, only the bit-flip (σ^x) for X is subject to error, the heat bath is in thermal equilibrium at the onset, and the fundamental Hamiltonian at the time is that the heat bath is $H_{\mathcal{N}} = \sum_k \hbar \omega_k a_k^{\dagger} a_k$, and the interaction Hamiltoniann is

$$H_{int} = \sum_j \sigma_j^x \otimes \sum_{\{\mathbf{k}\}} |k|^r \frac{\lambda}{\sqrt{M}} (e^{ikR_j} a_k + e^{-ikR_j} a_k^{\dagger}) \tag{11}$$

R_j means the spatial position of the qubit, M is the total number of modes, $\mathbf{k}$ is the wave number vector of modes, $r = 0, \pm 1/2$. Here we introduce the analysis by Hutter-Loss [6]. Let us consider how a given jth qubit evolves as it interacts with the heat bath. If the initial state of the system is $\rho_S \otimes \rho_{\mathcal{N}}$, then its decoherence evolution is expressed as follows using the operation according to the Lindblad equation.

$$\rho_S \longmapsto \Phi_e(\rho_S) = Tr_{\mathcal{N}}\{e^{-iHt}(\rho_S \otimes \rho_{\mathcal{N}})e^{iHt}\} \tag{12}$$

ρ_S is a density operator of all signal systems connected by correlation. We add the operation in the measurements of Stabilizer code, such as

$$\Psi_{\Pi}(\sigma) = \sum_a \Pi_a \sigma \Pi_a \tag{13}$$

Then we have the density operator for jth qubit as follows:

$$\begin{aligned} \rho_j(t) &= Tr_{\{k \neq j\}} \circ \Psi_{\Pi} \circ \Psi_e(\rho_S) \\ &= (1 - \eta_j(t, N))\rho_j + \eta_j(t, N)\sigma_j^x \rho_j \sigma_j^x \end{aligned} \tag{14}$$

where $\eta_j(t, N)$ is the error probability of the jth single qubit in a population of N qubits. From this formula, we can derive the probability of error for each qubit as a function of N and time, taking into account the correlation with other qubits. That is, for a set with quantum correlations, the influence from all other qubits will result in the following properties.

$$\begin{aligned} \eta_j(t, N+1) = \cos^2(J_{1,N+1})\eta_j(t, N) + \\ \sin^2(J_{1,N+1})(1 - \eta_j(t, N)) \end{aligned} \tag{15}$$

where

$$J_{m,n} \sim \lambda^2 \int dk \frac{|k|^{2r}}{\omega_k^2} \times \cos(k(R_m - R_n))(\sin(\omega_k t) - \omega_k t) \quad (16)$$

The above property comes from the recurrence phenomenon, and the probability of error for jth qubit can be described by

$$p(error) = \eta_j(t, N) \equiv f(t, \eta^*, N) \quad (17)$$

When N is increased, the above error probability follows the equations (15),(16).

B. Collective decoherence effect

Here we introduce collective decoherence such as generalized Dicke super radiation given by Lemberger and Yavus [4][5]. Let N atoms of a two-level system be qubits. Then we discuss more general discussin than the standard assumption that the wavelength of the radiation field is longer than the size of the qubit population in the interaction with the continuous mode field. Even if we take the above assumption, the system can be in the super-radiant region. Here, applying the Wigner-Weisskopf theory, it is well known that the interacting Hamiltonian such as generalized Dicke super radiation is

$$H_{int} = -\sum_j \sum_n \hbar\kappa_n (a_n \sigma_+^j + a_n^\dagger \sigma_-^j) \quad (18)$$

$$\sigma_z^j = |e>^{jj}< e| - |g>^{jj}< g|, \\ \sigma_+^j = |e>^{jj}< g|, \sigma_-^j = |g>^{jj}< e| \quad (19)$$

where j is qubit number and n is the mode number. Initially, the qubit system is assumed to be superimposed and the field is assumed to be a vacuum. At this time, the initial state of the two coupled systems is

$$|\Psi(t=0)> = \sum_{m=0}^{2^N-1} c_{m,0}|m, 0> \quad (20)$$

From the Schrödinger equation in the extended Hilbert space of the coupled system, the time evolution is

$$|\Psi(t)> = \sum_{m=0}^{2^N-1} c_{m,0}(t)|e^{-i(N_m\omega_a)t}|m, 0> + \sum_j \sum_{m'}^{2^N-1} c_{m',n}(t)e^{-i(N_{m'}\omega_a+\nu_n)t}|m', 1_n> \quad (21)$$

For simplicity, among N qubits, let $N/2 + \bar{N}$ be the excited state and $N/2 - \bar{N}$ be the ground state. The equation of motion for the stochastic amplitude of the point of interest in the above equation is as follows [4,5].

$$\frac{dc_{m,0}}{dt} = -(\frac{\Gamma}{2} + \delta\omega)(\frac{N}{2} + \bar{N})(\frac{N}{2} - \bar{N} + 1)c_{m,0} \quad (22)$$

where Γ and δ are the single decay rate and Lam shift, respectively. From the above, the decay of the probability amplitude of the representative point of interest is given as follows [4,5].

$$|c_{m,0}(t)|^2 \sim |c_{m,0}(t=0)|^2 e^{-(N^2/4)\Gamma t} \quad (23)$$

The above equation is applicable to the majority of stochastic amplitudes and it represents a feature of super-radiance.

Since super-radiance implies the simultaneous decay of the majority of qubits, one can next consider the non-locality of this super-radiance and analyze how the error probability of only certain qubits is affected by other qubits. In order to make the features easier to see, the initial state is set as follows.

$$|\psi(t=0)> = \sum_{m=0}^{2^{N-1}-1} c_{m,0}|g>^j \otimes|m, 0> + \sum_{m=0}^{2^{N-1}-1} d_{m,0}|e>^j \otimes|m, 0> \quad (24)$$

where m corresponds to an indicater of the quantum state of a qubit of $N-1$ other than jth qubit. If the density operator on the composit space is $\rho = |\psi><\psi|$, then the density operator of jth qubit is obtained by tracing this density operator over a qubit fraction of $N-1$. The result is

$$\rho_j = \sum_{m=0}^{2^{N-1}-1} |c_{m,0}|^2|g>^{jj}< g| + \sum_{m=0}^{2^{N-1}-1} |d_{m,0}|^2|e>^{jj}< e| + \sum_{m=0}^{2^{N-1}-1} c_{m,0}d_{m,0}^*|g>^{jj}< e| + \sum_{m=0}^{2^{N-1}-1} c_{m,0}^* d_{m,0}|e>^{jj}< g| \quad (25)$$

In the initial state, if the qubits of j are excited, N qubits radiate at once. On the other hand, if it is on the grand, the qubits of $N-1$ radiate at the same time. As a result.

$$|c_{m,0}(t)|^2 \sim |c_{m,0}(t=0)|^2 e^{-((N-1)^2/4)\Gamma t} \quad (26)$$

$$|d_{m,0}(t)|^2 \sim |d_{m,0}(t=0)|^2 e^{-(N^2/4)\Gamma t} \quad (27)$$

Considering the above equation in the equation (25), the longitudinal and transverse relaxation rates of the qubit of jth are

$$\frac{1}{T_1} \sim \frac{1}{T_2} \sim \Gamma N^\alpha \quad (28)$$

From the above, the error probability of the jth qubit at gate time δt is

$$p(error) = \eta_j(N) = f(\eta^*, N) \sim \Gamma N^\alpha \delta t \equiv \eta^* N^\alpha \quad (29)$$

This phenomenon is non-local, and the population decoherence causes a correlation error to the whole system with the same probability as in the above equation.

C. Leak from decoherence free subspace

Here we give a definition of decoherence free subspace (DFS) [12]. In a system that interacts with a heat bath, the evolution equation for the density operator that is taken the partial trace with respect to the heat bath is the Lindblad equation of the equation (8). Let the Hilbert space of the system of quantum bits be $\mathcal{H}_S$ and all density operators on it be $D(\mathcal{H}_S)$.
Definition: a decoherence free subspace : $\mathcal{H}_{DFS}$ is a subspace of $\mathcal{H}_S$ in which all density operators $\rho \in D(\mathcal{H}_{DFS})$ defined in that space satisfy the following equation.

$$\frac{\partial \rho}{\partial t} = \frac{-i}{\hbar}[H, \rho] \quad \forall t \tag{30}$$

In other words, it is equivalent to the absence of the effect of the Lindblad operator in the equation (8). The quantum states of the N qubits coupled in this space are given by the tensor product of the singlet state as follows.

$$|\Psi>_{DFS} = (\frac{1}{\sqrt{2}})^{N/2} \otimes_{j=1}^{N/2} (|g>|e> - |e>|g>)_j \tag{31}$$

If an error occurs in one of the quantum bits, the phenomenon in the equation (23) occurs. The decay rate of the stochastic amplitude at this time is interpreted as the rate at which the system leaks from the decoherence free subspace into a large extended Hilbert space [4,5]. As a result, the leak probability is regarded as follows.

$$p(Leak) \sim \Gamma N^2 \delta t \tag{32}$$

This is also nonlinear error, because the error probability depends on the number of qubit.

V. Information Theoretic Model Analysis of Quantum Errors

A. A model of quantum bit array structure

In the above sections, we have explained the serious phenomena on quantum noise in quantum computer. When a group of qubits is placed in a given environment, an increase in the number of qubits enhances the probability of error of one of its components. Here, we attempt to describe such quantum phenomena using only information-theoretic concepts [13], leaving out the physical processes. In this case, we can think of a qubit as just a bit, and model a two-dimensional arrangement of bits $x_{i,j}$ and the interaction of error factors $e_{i,j}$ from the environment with the qubits as follows.

$$\begin{pmatrix} x_{(1,1)} \bigoplus e_{(1,1)}, \dots x_{(1,L)} \bigoplus e_{(1,L)} \\ x_{(2,1)} \bigoplus e_{(2,1)}, \dots x_{(2,L)} \bigoplus e_{(2,L)} \\ \vdots \dots \\ x_{(L,1)} \bigoplus e_{(L,1)}, \dots x_{(L,L)} \bigoplus e_{(L,L)} \end{pmatrix} \tag{33}$$

where $x_{(i,j)}$ is the information bit of the spatial position (i, j) and $e_{(i,j)}$ is the error bit for that information bit. Here we emphasize the fact that only the probabilistic nature of the error is essential factor in the information theory. In the following, we devote to analyse such issues.

B. Nonlinear and local correlation errors by recurrence phenomena

If there is only one quantum bit, the probability of an error occurring in that qubit is η^*. Suppose here that $N_{sub1} = 5$ of qubits are set in a region. Let the latent probability of an error-induced in pairwise with the center and one of four qubits be $0 \leq p_1^* \leq 1/2$. In this case, the error probability of the central qubit (j) with subset (N_{sub1}) is given by the following:

$$\begin{aligned} \eta_j(N_{sub1}) &= \eta^* \sum_{q:even} \frac{4!}{q!(4-q)!}(p_1^*)^q(1-p_1^*)^{4-q} \\ &+(1-\eta^*) \sum_{q:odd} \frac{4!}{q!(4-q)!}(p_1^*)^q(1-p_1^*)^{4-q} \\ &= \frac{1}{2} - \frac{1}{2}(1-2\eta^*)(1-2p_1^*)^4 \\ &= \frac{1}{2} - \frac{1}{2}(1-2\eta^*)\Lambda_1 \end{aligned} \tag{34}$$

where $\Lambda_1 = (1 - 2p_1^*)$. From the recurrence phenomena, when four new qubits are set, the probability of an error for a single qubit in the center takes the initial probability given by the equation (34) from η^*. Here we set $\eta(N_{sub1}) = \eta_1^*$, and p_1 is replaced by p_2. Then we get the following

$$\begin{aligned} \eta_j(N_{sub2}) &= \eta_1^* \sum_{q:even} \frac{4!}{q!(4-q)!}(p_2^*)^q(1-p_2^*)^{4-q} \\ &+(1-\eta_1^*) \sum_{q:odd} \frac{4!}{q!(4-q)!}(p_2^*)^q(1-p_2^*)^{4-q} \\ &= \frac{1}{2} - \frac{1}{2}(1-2\eta_1^*)(1-2p_2^*)^4 \end{aligned} \tag{35}$$

From the equation (34) and $\Lambda_2 = (1 - 2p_2^*)^4$, the above becomes as follows:

$$\eta_j(N_{sub2}) = \frac{1}{2} - \frac{1}{2}(1-2\eta^*)\Lambda_1\Lambda_2 \tag{36}$$

In addition, if we substitute the initial probability into the same formula as the above formula, we get the following

$$p(error) = \eta_j(N_{subK}) = \frac{1}{2} - \frac{1}{2}(1-2\eta^*)\prod_{l=1}^{K}\Lambda_l \tag{37}$$

where $\Lambda_l = (1 - 2p_l^*)^4$, and $l = \{1, 2, \dots, K\}$.

Thus, as the number of qubits increases, the error probability of each qubit alone increases. Thus this information theoretic modeling can visualize the following physical phenomena. Despite the nature of the quantum noise from the environment is being invariant, the probability of its own error increases when qubits are clustered together. As a special case, we have

$$p(error) = \eta_j(N_{subK}) \quad = \quad \eta^* \quad p_l^* = 0 \quad \forall l \tag{38}$$

$$p(error) = \eta_j(N_{subK}) \quad \rightarrow \quad \frac{1}{2} \quad p_l^* \neq 0, N > 1 \tag{39}$$

In the model shown here, if the propagation of the error does not happen to occur in other qubits, it is a simple nonlinear independent error, and if the error propagates to all qubits connected by quantum correlations between neighboring qubits, it is a nonlinear and local correlation error.

C. Nonlinear and non-local correlation errors by collective decoherence

Let us discuss first the nonlinear effect. When the jth qubit in some space interacts with one other qubit, let the collapse rate of jth qubit due to the effect of that interaction be $\chi_{j,k}$. The error probability of jth qubit due to its interaction is a function of $\chi_{j,k}$:

$$p(error) = f(\chi_{j,k}) \tag{40}$$

Then the collapse rate of the jth qubit due to the interaction between the jth and all the existing qubits is the sum of each collapse rate as follows:

$$\chi_j(N) = \sum_{k=1}^{N-1} \chi_{j,k} \tag{41}$$

In this case, the error probability of a single jth qubit in a N qubits system is

$$p(error) = \eta_j(N) = f(\sum_{k=1}^{N-1} \chi_{j,k}) = f(\chi_j(N)) \tag{42}$$

The specific form of the above equation depends on physical phenomena, but it can be approximated and considered in the following form.

$$p(error) = \eta_j(N) \cong \eta^* N^{\alpha} \tag{43}$$

The above model gives a different nonlinear effect in the error performance of that in the former section. In the above discussins, we have asuumed that the whole qubits have a correlation based on the collective decoherence theory such as Dicke super radiation.

Then we have the clusterd noise effect by such collective decoherence, and it stimulates the burst error, in which the whole qubits are destroyed simultaneously with the following probability from the equation (43):

$$P_e(N) = \eta_j(N) \cong \eta^* N^{\alpha} \tag{44}$$

Thus, from our formulation, it is clear that the conventional quantum error correction code cannot hold the function of error correction for these error phenomena.

VI. Countermeasure Research

We have explained new quantum noise phenomena which cause a serious error in quantum computers. In general, these quantum error cannot be eliminated by the conventional design theory. In order to cope with the quantum errors described in this paper, or to break this situation, one way is to establish a new way to physically suppress such error, and to further develop the conventional quantum error correction theory [14,15] based on the quantum noise analysis. Both are extremely challenging problems.

VII. Conclusion

The amazing capabilities of quantum computers are made possible by making the best use of the features of quantum mechanics. However, in real-world environments, performance limits are caused by noise, but the properties of that noise have been treated as a quantum version of classical noise. If the computational process makes use of all quantum effects, then we should also assume noise with all quantum effects. However, so far the very simple quantum noise such as Pauli noise has been modeled. In this paper, we have clarified how to model all possible quantum noise in an information-theoretic manner. Especialy, the nonlinear error by the recurrence effect of entanglement and the collective decoherence have been analyzed, which has no correspondence in the classical computers. In the next paper, we will investigate a synchronization quantum error based on Belavkin equation [11] or quantum van der Pole equation [16][17]

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References

[1] National Academies of Sciences, Engineering, and Medicine, “Quantum Computing: Progress and Prospects”, Edited by Emily Grumbling, Mark Horowitz, 2019.
[2] National Institute of Standards and Technology, “Status Report on the Second Round of the NIST Post-Quantum Cryptography Standardization Processes”, National Institute of Standards and Technology Interagency or Internal Report 8309, 2020.
[3] J.Preskil, “Sufficient condition on noise correlations for scalable quant. comput.”, Quant. Inf. Comput., 13, 181, 2013.
[4] B.Lemberger, and D.D.Yavus, “Effect of correlated decay on fault tolerant quantum computation”, Physical Review A. vol-96, 062337, 2017.
[5] D.D.Yavus, “Superradiance as a source of collective decoherence in quantum computer”, J. Opt. Soc.Am., B31, 2665,2014.
[6] A.Hutter and D.Loss, “Breackdown of surface code error correction due to coupling to a bosonic bath”, Phys. Rev. A. vol-89, 042334, 2014.
[7] A.G.Fowler, M.Mariantoni, J.M.Martinis, and A.N.Cleland, Surface codes: “Towards practical large-scale quantum computation”, Phys. Rev. A 86, 032324 2012.
[8] D.Staudt, The role of correlated noise in quantum computing, arXiv:1111.1417. 2011.
[9] O.Hirota, “The current state of quantum computers and their relationship with cryptographic research”, Japan Security Summit, December 17, 2019.
[10] O.Hirota, “Quantum noise analysis for quantum computer”, The IEICE Technical Report on Information Theory, IT-2020-17, pp1-6, July, 2020.
[11] V.P.Belavkin, O.Hirota, and R.L.Hudson:, “The world of quantum noise and the fundamental outout processes”, Proc. of Qauntum Communication and Measurement”, Plenum Press(Springer) , 1995.
[12] R.Karasik, K.Marzlin, B.Sanders, K.Whaley, “Multiparticle decoherence free subspaces in extended systems,” *Physical Review A*, vol. 76, 012331, 2007.
[13] T.M.Cover, J.A.Thomas, “Elements of information theory”, John Wiley and Sons, 2006.
[14] J.Kempe, “Approaches to quantum error correcction”, Poincare Seminar, 1, pp65-93, 2005.
[15] Ivan Djordjevic, “Quantum Information Processing and Quantum Error Correction: An Engineering Approach”, Academic Press, 2012.

[16] T.S.Usuda, O.Hirota, “An optical receiver overcome the standard quantum limit”, The IEICE Trans. Communications, vol-E75 B, pp514-520, 1992.
[17] O.Hirota, Y.Suematsu,“Noise properties of injection lasers due to reflected waves”, IEEE Journal of Quantum Electronoics, vol-QE-15, pp142-149, 1979.