# Detection of the binary phase-encoded quasi-Bell state signal in a lossy environment by a half beam splitter and photon counters

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## Detection of the binary phase-encoded quasi-Bell state signal in a lossy environment by a half beam splitter and photon counters

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*Abstract*—A simple detection scheme for a particular pair of the quasi-Bell states was proposed in the literature [8]. This detection scheme consists of a half beam splitter and photon counters and can realize error-free detection when the states are pure [7]. In this paper, the performance of this detection scheme in a lossy environment is analyzed.

#### I. INTRODUCTION

The term "entangled coherent state" was coined by Sanders [1], [2]. Nowadays, a family of entangled coherent states is widely recognized as a potential resource for quantum information processing [3]. The purpose of our research is to develop and improve the coherent statebased quantum information and communication technologies. In this paper, we focus on the so-called quasi-Bell states of entangled coherent states.

The quasi-Bell states were introduced in the literature [4]. The first remarkable application of the quasi-Bell states is found in the study of quantum teleportation [5], [6]. Through the study of quantum reading, Hirota pointed out that a particular pair of the quasi-Bell states,  $|\alpha\rangle|\pm\alpha\rangle - |-\alpha\rangle|\mp\alpha\rangle$ , can provide error-free detection [7]. To implement such error-free detection, a simple detection scheme was proposed in the literature [8]. The proposed detection scheme is shown in Fig.1, which consists of a half beam splitter (HBS) and two photon counters (PCs). In the literature [8], this scheme was investigated only in the noiseless case, while the corresponding optimal quantum receiver (the minimax quantum receiver) was investigated not only in the noiseless case but also in the case of a lossy environment. Therefore, the analysis of the proposed detection scheme in a lossy environment is still remaining. In this paper, we analyze such a case.

#### II. DETECTION BY HBS AND PCS

Suppose an entangled coherent state

$$|H\rangle_{AB} = \mathcal{N}(|\alpha\rangle_{A}|\alpha\rangle_{B} - |-\alpha\rangle_{A}|-\alpha\rangle_{B})$$
(1)

is prepared as a resource for information processing, where  $\mathcal{N} = 1/\sqrt{2(1-\kappa^2)}$  and  $\kappa = \langle \alpha | -\alpha \rangle = \exp[-2|\alpha|^2]$ . This state can be generated



Fig. 1. A detection scheme [8]

by a cat state  $|cat\rangle = \mathcal{N}(|\sqrt{2}\alpha\rangle_{A} - |-\sqrt{2}\alpha\rangle_{A})$ and a HBS characterized [10] by  $|\alpha'\rangle_{A}|\beta'\rangle_{B} \rightarrow |(\alpha' + \beta')/\sqrt{2}\rangle_{A}|(\alpha' - \beta')/\sqrt{2}\rangle_{B}$ . The basic idea of this state generation technique was introduced by Yurke and Stoler [9].

For the noiseless case, one can consider the following binary states by phase-encoding for light in mode B of the state  $|H\rangle_{AB}$ :

$$\begin{aligned} |\text{signal } 0\rangle_{AB} &= \mathcal{N} \left( |\alpha\rangle_{A} |\alpha\rangle_{B} - |-\alpha\rangle_{A} |-\alpha\rangle_{B} \right), (2) \\ |\text{signal } 1\rangle_{AB} &= \mathcal{N} \left( |\alpha\rangle_{A} |-\alpha\rangle_{B} - |-\alpha\rangle_{A} |\alpha\rangle_{B} \right). (3) \end{aligned}$$

These two states — which are members of the quasi-Bell states and form a binary phase-encoded signal — are orthogonal, and hence the average probability of detection error can be zero [7].

Here we suppose optical paths between the light source and the detector are placed in a lossy environment. As shown in Fig. 1, output lights from the light source are traveling through the optical paths characterized by the loss parameters  $\epsilon_A$  and  $\epsilon_B$ , respectively. When the light of mode B reached to the modulator, it is phase-encoded in accordance with classical data bit 0/1, while the light of mode A is reflected at the mirror. After that, each light is traveling through a lossy path having loss parameter  $\epsilon'_A$ or  $\epsilon'_B$  towards the detectior. Therefore, the total loss of each path between the light source to the detector is given by  $\eta_A = \epsilon_A \epsilon'_A$  or  $\eta_B = \epsilon_B \epsilon'_B$ . Then the possible signal states in front of the detector are expressed as follows [8]:

$$\begin{aligned} \hat{\boldsymbol{\rho}}_{0}^{\text{signal}} \\ &= \mathcal{N}^{2} \\ &\times |\alpha\sqrt{\eta_{A}}\rangle_{A} \langle \alpha\sqrt{\eta_{A}}| \otimes |\alpha\sqrt{\eta_{B}}\rangle_{B} \langle \alpha\sqrt{\eta_{B}}| \\ &-\mathcal{N}^{2} \kappa^{1-\eta_{A}} \kappa^{1-\eta_{B}} \\ &\times |\alpha\sqrt{\eta_{A}}\rangle_{A} \langle -\alpha\sqrt{\eta_{A}}| \otimes |\alpha\sqrt{\eta_{B}}\rangle_{B} \langle -\alpha\sqrt{\eta_{B}}| \\ &-\mathcal{N}^{2} \kappa^{1-\eta_{A}} \kappa^{1-\eta_{B}} \\ &\times |-\alpha\sqrt{\eta_{A}}\rangle_{A} \langle \alpha\sqrt{\eta_{A}}| \otimes |-\alpha\sqrt{\eta_{B}}\rangle_{B} \langle \alpha\sqrt{\eta_{B}}| \\ &+\mathcal{N}^{2} \\ &\times |-\alpha\sqrt{\eta_{A}}\rangle_{A} \langle -\alpha\sqrt{\eta_{A}}| \otimes |-\alpha\sqrt{\eta_{B}}\rangle_{B} \langle -\alpha\sqrt{\eta_{B}}| \end{aligned}$$
(4)

and

$$\begin{aligned} \hat{\rho}_{1}^{\text{signal}} &= \mathcal{N}^{2} \\ &\times |\alpha \sqrt{\eta_{A}}\rangle_{A} \langle \alpha \sqrt{\eta_{A}} | \otimes |-\alpha \sqrt{\eta_{B}}\rangle_{B} \langle -\alpha \sqrt{\eta_{B}} | \\ &- \mathcal{N}^{2} \kappa^{1-\eta_{A}} \kappa^{1-\eta_{B}} \\ &\times |\alpha \sqrt{\eta_{A}}\rangle_{A} \langle -\alpha \sqrt{\eta_{A}} | \otimes |-\alpha \sqrt{\eta_{B}}\rangle_{B} \langle \alpha \sqrt{\eta_{B}} | \\ &- \mathcal{N}^{2} \kappa^{1-\eta_{A}} \kappa^{1-\eta_{B}} \\ &\times |-\alpha \sqrt{\eta_{A}}\rangle_{A} \langle \alpha \sqrt{\eta_{A}} | \otimes |\alpha \sqrt{\eta_{B}}\rangle_{B} \langle -\alpha \sqrt{\eta_{B}} | \\ &+ \mathcal{N}^{2} \\ &\times |-\alpha \sqrt{\eta_{A}}\rangle_{A} \langle -\alpha \sqrt{\eta_{A}} | \otimes |\alpha \sqrt{\eta_{B}}\rangle_{B} \langle \alpha \sqrt{\eta_{B}} |. \end{aligned}$$
(5)

For each state, the average numbers of signal photons of each mode are given by

$$\langle \hat{n}_{A} \rangle |_{\text{signal},0} = \langle \hat{n}_{A} \rangle |_{\text{signal},1}$$
  
=  $2\mathcal{N}^{2} |\alpha|^{2} \eta_{A} (1+\kappa^{2})$  (6)

and

$$\begin{aligned} \langle \hat{n}_{\rm B} \rangle |_{\rm signal,0} &= \langle \hat{n}_{\rm B} \rangle |_{\rm signal,1} \\ &= 2\mathcal{N}^2 |\alpha|^2 \eta_{\rm B} (1+\kappa^2). \end{aligned} \tag{7}$$

When the minimax quantum receiver is employed for discriminating the signal states, the average probability of error is given by [8]

$$\bar{P}_{\rm e}^{\rm minmax} = \frac{1}{2} \left( 1 - \sqrt{\frac{1 - \kappa^{2\eta_{\rm A}}}{1 - \kappa^2}} \sqrt{\frac{1 - \kappa^{2\eta_{\rm B}}}{1 - \kappa^2}} \right).(8)$$

In particular, when  $\eta_{\rm A} = \eta_{\rm B}$ , it becomes [8]

$$\bar{P}_{\rm e}^{\rm minmax}\big|_{\eta_{\rm A}=\eta_{\rm B}} = \frac{\kappa^{2\eta_{\rm A}}-\kappa^2}{2(1-\kappa^2)}.$$
 (9)

#### III. PHOTON DISTRIBUTION AT PCs

Here we suppose the proposed detection scheme, which is illustrated in Fig. 1, is employed for discriminating the signal states. In this detection scheme, a two-mode light in the state  $\hat{\rho}_0^{\text{signal}}$  or  $\hat{\rho}_1^{\text{signal}}$  is entering the HBS of the detector first. The possible states that appear after passing through the HBS of the detector and in front of the PCs of the detector are given as follows:

$$\hat{\boldsymbol{\rho}}_{0} = \mathcal{N}^{2} |\sigma\rangle_{A} \langle\sigma| \otimes |\tau\rangle_{B} \langle\tau| \\
-\mathcal{N}^{2} \kappa^{1-\eta_{A}} \kappa^{1-\eta_{B}} |\sigma\rangle_{A} \langle-\sigma| \otimes |\tau\rangle_{B} \langle-\tau| \\
-\mathcal{N}^{2} \kappa^{1-\eta_{A}} \kappa^{1-\eta_{B}} |-\sigma\rangle_{A} \langle\sigma| \otimes |-\tau\rangle_{B} \langle\tau| \\
+\mathcal{N}^{2} |-\sigma\rangle_{A} \langle-\sigma| \otimes |-\tau\rangle_{B} \langle-\tau|$$
(10)

and

$$\hat{\boldsymbol{\rho}}_{1} = \mathcal{N}^{2} |\tau\rangle_{A} \langle \tau | \otimes |\sigma\rangle_{B} \langle \sigma | 
-\mathcal{N}^{2} \kappa^{1-\eta_{A}} \kappa^{1-\eta_{B}} |\tau\rangle_{A} \langle -\tau | \otimes |\sigma\rangle_{B} \langle -\sigma | 
-\mathcal{N}^{2} \kappa^{1-\eta_{A}} \kappa^{1-\eta_{B}} |-\tau\rangle_{A} \langle \tau | \otimes |-\sigma\rangle_{B} \langle \sigma | 
+\mathcal{N}^{2} |-\tau\rangle_{A} \langle -\tau | \otimes |-\sigma\rangle_{B} \langle -\sigma |,$$
(11)

where the complex amplitudes  $\sigma$  and  $\tau$  of component coherent states are

$$\sigma = \frac{1}{\sqrt{2}}\alpha(\sqrt{\eta_{\rm A}} + \sqrt{\eta_{\rm B}}), \qquad (12)$$

$$\tau = \frac{1}{\sqrt{2}}\alpha(\sqrt{\eta_{\rm A}} - \sqrt{\eta_{\rm B}}). \tag{13}$$

The average numbers of photons at PCs are respectively given by

$$\begin{aligned} \langle \hat{n}_{A} \rangle |_{\text{for } \hat{\boldsymbol{\rho}}_{0}} &= \langle \hat{n}_{B} \rangle |_{\text{for } \hat{\boldsymbol{\rho}}_{1}} \\ &= \mathcal{N}^{2} |\alpha|^{2} \left( \sqrt{\eta_{A}} + \sqrt{\eta_{B}} \right)^{2} \left( 1 + \kappa^{2} \right) \end{aligned}$$
(14)

and

$$\begin{aligned} \langle \hat{n}_{\rm B} \rangle |_{\text{for } \hat{\boldsymbol{\rho}}_0} &= \langle \hat{n}_{\rm A} \rangle |_{\text{for } \hat{\boldsymbol{\rho}}_1} \\ &= \mathcal{N}^2 |\alpha|^2 \left(\sqrt{\eta_{\rm A}} - \sqrt{\eta_{\rm B}}\right)^2 (1 + \kappa^2), \end{aligned} \tag{15}$$

and the photon distributions at PCs are respectively given by

$$P(m,n|0) = {}_{A}\langle m|_{B}\langle n|\hat{\rho}_{0}|m\rangle_{A}|n\rangle_{B}$$
  
$$= 2\mathcal{N}^{2} \exp[-R]$$
  
$$\times \left\{1 - \kappa^{1-\eta_{A}}\kappa^{1-\eta_{B}}(-1)^{m}(-1)^{n}\right\}$$
  
$$\times \frac{S^{m}}{m!} \cdot \frac{T^{n}}{n!}$$
(16)

and

$$P(m,n|1) = {}_{\mathrm{A}}\langle m|_{\mathrm{B}}\langle n|\hat{\rho}_{1}|m\rangle_{\mathrm{A}}|n\rangle_{\mathrm{B}}$$
  
$$= 2\mathcal{N}^{2} \exp[-R]$$
  
$$\times \left\{1 - \kappa^{1-\eta_{\mathrm{A}}} \kappa^{1-\eta_{\mathrm{B}}} (-1)^{m} (-1)^{n}\right\}$$
  
$$\times \frac{T^{m}}{m!} \cdot \frac{S^{n}}{n!}, \qquad (17)$$

where

$$R = |\alpha|^2 (\eta_{\rm A} + \eta_{\rm B}), \qquad (18)$$

$$S = \frac{1}{2} |\alpha|^2 (\sqrt{\eta_{\rm A}} + \sqrt{\eta_{\rm B}})^2,$$
 (19)

$$T = \frac{1}{2} |\alpha|^2 \left(\sqrt{\eta_{\rm A}} - \sqrt{\eta_{\rm B}}\right)^2, \qquad (20)$$

and where the convention  $0^0 = 1$  has been used.

When  $\eta_A = \eta_B$ , the two distributions of Eqs.(16) and (17) overlap only at the case m = n = 0. This is illustrated in Fig. 2. When  $\eta_A \neq \eta_B$ , the overlap of the two photon distributions is spread over all region. This is illustrated in Fig. 3, where negligibly small probabilities (< 10<sup>-12</sup>) are omitted.

#### IV. PERFORMANCE ANALYSIS

First, we assume  $\eta_{\rm A} = \eta_{\rm B}$ . In this case, the photon distributions are reduced to the following forms.

$$P(m, n|0)|_{\eta_{A}=\eta_{B}} = \begin{cases} \frac{\kappa^{2\eta_{A}} - \kappa^{2}(-1)^{m}}{\kappa^{\eta_{A}}(1-\kappa^{2})} \times \frac{S^{m}}{m!}, & \text{for } n = 0; \\ 0, & \text{for } n \neq 0 \end{cases}$$
(21)

and

$$= \begin{cases} P(m, n|1)|_{\eta_{A}=\eta_{B}} \\ \frac{\kappa^{2\eta_{A}} - \kappa^{2}(-1)^{n}}{\kappa^{\eta_{A}}(1-\kappa^{2})} \times \frac{S^{n}}{n!}, & \text{for } m = 0; \\ 0, & \text{for } m \neq 0. \end{cases}$$
(22)

Since the two distributions overlap only at the case m = n = 0, the average probability of error is given by

$$\bar{P}_{\rm e}\big|_{\eta_{\rm A}=\eta_{\rm B}} = \frac{1}{\kappa^{\eta_{\rm A}}} \times \frac{\kappa^{2\eta_{\rm A}} - \kappa^2}{2(1-\kappa^2)},$$
 (23)

where the classical minimax criterion has been used for determining the decision regions. A quantitative behavior of the average probability of error for the case of  $\eta_A = \eta_B$  is shown in Fig. 4. Comparing  $\bar{P}_e$  of Eq. (23) with  $\bar{P}_e^{\rm minimax}$  of Eq. (9), we see that  $\bar{P}_e$  is rapidly degraded in accordance with loss parameter  $\eta_A$  bacause of the factor  $1/\kappa^{\eta_A}$ .

Next, we assume  $\eta_A \neq \eta_B$ . In this case, the overlap of the two disctributions is spread to all region. Note that P(m, n|0) and P(m, n|1) are symmetric with respect to m and n, that is, P(m, n|0) = P(n, m|1). The likelihood of the two distributions is

$$\frac{P(m,n|1)}{P(m,n|0)}\Big|_{\eta_{\mathrm{A}}\neq\eta_{\mathrm{B}}} = \left(\frac{\sqrt{\eta_{\mathrm{A}}} - \sqrt{\eta_{\mathrm{B}}}}{\sqrt{\eta_{\mathrm{A}}} + \sqrt{\eta_{\mathrm{B}}}}\right)^{2(m-n)},\quad(24)$$

and hence the average probability of error is given by

$$\bar{P}_{\rm e}\big|_{\eta_{\rm A} \neq \eta_{\rm B}} = \sum_{n} \sum_{m < n} P(m, n|0) + \frac{1}{2} \sum_{n} P(n, n|0),$$
(25)

where the relation  $(\sqrt{\eta_{\rm A}} - \sqrt{\eta_{\rm B}})^2 \leq (\sqrt{\eta_{\rm A}} + \sqrt{\eta_{\rm B}})^2$ has been used for determining the threshold line in accordance with the classical minimax criterion. Figure 5 shows a typical numerical behavior of  $\bar{P}_{\rm e}$ , together with the corresponging error probability by the minimax quantum receiver. From this, we see that the performance is affected by the difference of loss parameters,  $\sqrt{\eta_{\rm A}} - \sqrt{\eta_{\rm B}}$ . A large difference of the loss paremeters degrades the performance.

#### V. CONCLUSION

A simple detection scheme proposed in the literature [8] for the binary phase-encoded quasi-Bell state signal was analyzed in the case of a lossy environment. From the performance analysis above, it was clarified that the proposed detection scheme is fairly sensitive to the loss parameters of optical paths from the light source to the detector, despite that it can realize error-free detection for the noiseless case.

In the sense of the minimum error criterion, the proposed scheme does not indicate better performance. However, it shows an interesting property when the loss parameters of the optical paths are identical. In that case, the photon distributions overlap only at the outcome n = m = 0. This property may offer other useful applications of the proposed detection scheme, because it can be regarded as error-free detection by discarding the overlap outcome. We will discuss this problem elsewhere.

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Fig. 2. [color online] P(m, n|0) — blue — and P(m, n|1) — orange — at  $\alpha = 2$ . (i)  $\eta_{\rm A} = \eta_{\rm B} = 1.0$ . (ii)  $\eta_{\rm A} = \eta_{\rm B} = 0.9$ . (iii)  $\eta_{\rm A} = \eta_{\rm B} = 0.5$ . (iv)  $\eta_{\rm A} = \eta_{\rm B} = 0.1$ .

Fig. 3. [color online] P(m, n|0) — blue — and P(m, n|1) — orange — at  $\alpha = 2$ . (i)  $\eta_A = 0.6$  and  $\eta_B = 0.4$ . (ii)  $\eta_A = 0.7$  and  $\eta_B = 0.3$ . (iii)  $\eta_A = 0.8$  and  $\eta_B = 0.2$ . (iv)  $\eta_A = 0.9$  and  $\eta_B = 0.1$ .





Fig. 5. [color online]  $\bar{P}_{\rm e}$  versus  $\bar{n}_{\rm A} + \bar{n}_{\rm B}$  when  $\eta_{\rm A} \neq \eta_{\rm B}$ . Red line stands for  $\bar{P}_{\rm e}$  of Eq. (25) and blue dashed line for  $\bar{P}_{\rm e}^{\rm minimax}$  of Eq. (8). (i)  $\eta_{\rm A} = 0.6$  and  $\eta_{\rm B} = 0.4$ . (ii)  $\eta_{\rm A} = 0.7$  and  $\eta_{\rm B} = 0.3$ . (iii)  $\eta_{\rm A} = 0.8$  and  $\eta_{\rm B} = 0.2$ . (iv)  $\eta_{\rm A} = 0.9$  and  $\eta_{\rm B} = 0.1$ .

Fig. 4. [color online]  $\bar{P}_{\rm e}$  versus  $\bar{n}_{\rm A} + \bar{n}_{\rm B}$  when  $\eta_{\rm A} = \eta_{\rm B}$ . Red line stands for  $\bar{P}_{\rm e}$  of Eq. (23) and blue dashed line for  $\bar{P}_{\rm e}^{\rm minimax}$  of Eq. (8). The average number of photons is given by Eqs. (14) and (15). (i)  $\eta_{\rm A} = \eta_{\rm B} = 0.999$ . (ii)  $\eta_{\rm A} = \eta_{\rm B} = 0.99$ . (iii)  $\eta_{\rm A} = \eta_{\rm B} = 0.9$ . (iv)  $\eta_{\rm A} = \eta_{\rm B} = 0.5$ . (v)  $\eta_{\rm A} = \eta_{\rm B} = 0.1$ .