ISSN 2186-6570

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Tamagawa University Quantum ICT Research Institute Bulletin, Vol.10, No.1, 23-28, 2020

Tamagawa University Quantum ICT Research Institute Bulletin
Vol.10 No.1 : 23–28（2020）

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***Abstract*—A quantum tester is a powerful tool for formulating quantum process discrimination problems, where a process is a generalization of a quantum channel. The goal of this paper is to introduce the basics of process discrimination problems using quantum testers. For an intuitive illustration, we will often use diagrammatic representations.**

I. Introduction

Discrimination of quantum channels is a fundamental problem in the field of quantum information theory. The aim of this work is to distinguish between a given finite set of known quantum channels as well as possible. Quantum state discrimination, which is its special case, has been extensively investigated [1]–[14], starting with the seminal works of Helstrom, Holevo, and Yuen *et al.* [15]–[17]. A more general quantum channel discrimination problem has also been investigated [18]–[30]; however, since it is usually harder to solve than a state discrimination problem, the properties of optimal discrimination are not known except for some special cases. In the case in which the channel can be used several times, optimal discrimination may be adaptive, which is usually very difficult to handle.

It is known that a quantum tester is a useful tool for expressing quantum channel discrimination problems mathematically and diagrammatically (e.g., [31], [32]). A quantum tester can be interpreted as an extension of a quantum measurement. In a quantum state discrimination problem, the task is to optimize a measurement; similarly, in a quantum channel discrimination problem, the task is to optimize a tester [33]. Any discrimination allowed by quantum theory, including the adaptive one, can be represented by a tester. In this paper, we deal with the discrimination of quantum processes, which are generalizations of quantum channels. Channels (with/without memory), subchannels, and processes consisting of multiple time steps are all special cases of processes.

Process discrimination problems can be mathematically described by using the Choi-Jamiołkowski representations of processes and testers. However, such representations may actually suffer from the lack of an intuitive grasp of the operational implications. In contrast, the diagrammatic representations of them enable us to get some intuitive insight almost without loss of representation power. In this paper, we introduce the basics of process discrimination problems using diagrammatic representations, which will help the readers in a better understanding of the properties of optimal testers derived, e.g., in Refs. [33], [34].

II. Quantum processes and testers

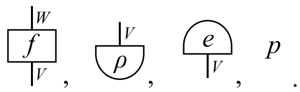
A. Preliminaries

First, we will give some notations. $\mathbf{R}_+$ and $\mathbf{C}$ denote, respectively, the sets of all nonnegative real numbers and complex numbers. Let N_V be the dimension of a system (i.e., a complex Hilbert space) V. In this paper, we restrict our attention to finite-dimensional systems. A one-dimensional system is identified with $\mathbf{C}$. Pos_V denotes the set of all positive semidefinite matrices on V. Also, Den_V stand for the set of all density matrices (i.e., positive semidefinite matrices with unit trace) on V. A positive semidefinite matrix is called pure if it has rank one. $\mathsf{Pos}(V, W)$ and $\mathsf{Chn}(V, W)$ stand for, respectively, the sets of all completely positive (CP) maps and all trace-preserving CP maps from Pos_V to Pos_W. By abuse of notation, we will assume $\mathsf{Chn}(\mathbf{C}, V) = \mathsf{Chn}(V, \mathbf{C}) = \mathsf{Den}_V$, $\mathsf{Pos}(\mathbf{C}, V) = \mathsf{Pos}(V, \mathbf{C}) = \mathsf{Pos}_V$, and $\mathsf{Pos}_\mathbf{C} = \mathbf{R}_+$. Let I_V and $\mathbb{1}_V$ be, respectively, the identity maps on V and Pos_V. Each single-step process is described by a CP map.

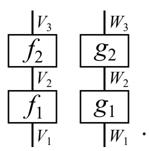
B. Diagrammatic representations

We attempt to provide an intuitive understanding of an operational interpretation by using diagrammatic representations. For details, we refer to the work of Coecke, Abramsky, and others (see, e.g., [35]–[37]; one can also see [38], [39]).

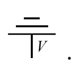
A quantum process is depicted by a combination of single-step processes. Each single-step process has input and output systems. A single-step process is called a state if its input system is $\mathbf{C}$. Similarly, a single-step process is called an effect if its output system is $\mathbf{C}$. A scalar is a single-step process on $\mathbf{C}$. In diagrammatic terms, a single-step process $f \in \mathsf{Pos}(V, W)$, a state $\rho \in \mathsf{Pos}(\mathbf{C}, V)$, an effect $e \in \mathsf{Pos}(V, \mathbf{C})$, and a scalar $p \in \mathbf{R}_+$ are depicted as



Single-step processes can be connected sequentially or in parallel. The sequential composition of $f \in \mathsf{Pos}(V_1, V_2)$ and $g \in \mathsf{Pos}(V_2, V_3)$ is also a process, denoted as $g \circ f \in \mathsf{Pos}(V_1, V_3)$. Similarly, the parallel composition of $f \in \mathsf{Pos}(V_1, V_2)$ and $h \in \mathsf{Pos}(W_1, W_2)$ is a process, denoted as $f \otimes h \in \mathsf{Pos}(V_1 \otimes W_1, V_2 \otimes W_2)$. For example, for any $f_1 \in \mathsf{Pos}(V_1, V_2)$, $f_2 \in \mathsf{Pos}(V_2, V_3)$, $g_1 \in \mathsf{Pos}(W_1, W_2)$, and $g_2 \in \mathsf{Pos}(W_2, W_3)$, $(f_2 \otimes g_2) \circ (f_1 \otimes g_1) = (f_2 \circ f_1) \otimes (g_2 \circ g_1)$ is diagrammatically depicted by

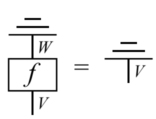


1) Discarding effects: The effect represented by $I_V \in \mathsf{Pos}(V, \mathbf{C})$[1] is called a discarding effect, which is denoted by $\top_V$, or simply $\top$. This is diagrammatically depicted as



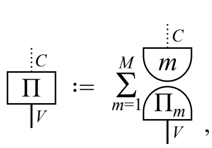
$\top_V$ intuitively means that one performs a measurement on system V and then discards its results.

2) Channels: A single-step process $f \in \mathsf{Pos}(V, W)$ is called a channel if

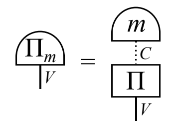


holds, which is equivalently written as $f \circ \rho \in \mathsf{Chn}_W$ for any $\rho \in \mathsf{Chn}_V$. f can be expressed by a trace-preserving CP map. Any normalized state (i.e., any state described by a density matrix) can be regarded as a special case of a channel.

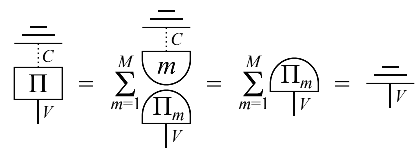
3) Measurements: A quantum measurement Π on V can be represented by a set of effects $\{\Pi_m \in \mathsf{Pos}_V\}_{m=1}^M$ satisfying $\sum_{m=1}^M \Pi_m = \top_V$, which is often called a positive operator-valued measure (POVM). In this paper, M often denotes the number of elements of Π. Π is also represented by



where C denotes a classical system, which is depicted as the dotted line, and the state m denotes $|m\rangle\langle m| \in \mathsf{Den}_C$ ($\{|1\rangle, \ldots, |M\rangle\}$ is the standard orthonormal basis of C). Each effect Π_m, which is called a POVM element, satisfies



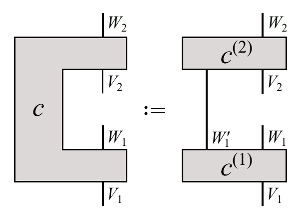
One can easily verify



i.e., $\Pi \in \mathsf{Chn}(V, C)$.

C. Quantum processes

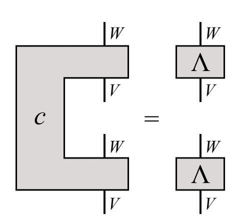
Let us consider a process, denoted by c, consisting of T single-step processes $\{c^{(t)} \in \mathsf{Pos}(W'_{t-1} \otimes V_t, W'_t \otimes W_t)\}_{t=1}^T$, where T is a natural number and $W'_0 = W'_T = \mathbf{C}$. For example, in the case of $T = 2$, c is diagrammatically depicted by


(1)

where $W'_1, \ldots, W'_{T-1}$ are the internal systems of process c. We will mathematically express this process c by

$$c := c^{(T)} \circledast c^{(T-1)} \circledast \cdots \circledast c^{(1)}, \tag{2}$$

where $\circledast$ denotes the connection of processes, which is called the link product [31]. Note that only two systems with the same label can be connected. For any t and t' with $t < t'$, $c^{(t')}$ is in the causal future of $c^{(t)}$, i.e., $c^{(t)}$ can signal to $c^{(t')}$ but not vice versa. A simple example of a process is $c = \Lambda \circledast \Lambda$, with $\Lambda \in \mathsf{Chn}(V, W)$, where $V_1 = V_2 = V$, $W_1 = W_2 = W$, and $W'_1 = \mathbf{C}$. This process is depicted as



Let

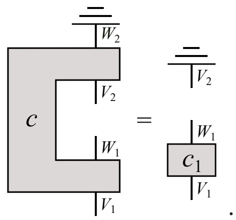
$$\circledast_{t=1}^T \mathsf{Pos}(V_t, W_t) := \mathsf{Pos}(V_T, W_T) \circledast \cdots \circledast \mathsf{Pos}(V_1, W_1)$$

be the set of all quantum processes expressed in the form of Eq. (2).

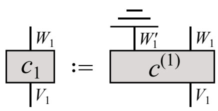
A process c is called a quantum comb if it is expressed in the form of Eq. (2) with channels $c^{(1)}, \ldots, c^{(T)}$. Let

[1] This can also be represented by the linear map $\mathsf{Pos}_V \ni \rho \mapsto \mathrm{Tr}\,\rho \in \mathbf{R}_+$.

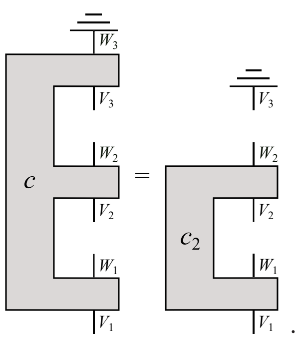
$\circledast_{t=1}^{T} \mathsf{Chn}(V_t, W_t)$ be the set of all quantum combs expressed in the form of Eq. (2). In the case of $T = 2$, for any comb c, there exists $c_1 \in \mathsf{Chn}(V_1, W_1)$ satisfying



Indeed, we can easily check it by letting



and using Eq. (1). The converse also holds [31]. Similarly, in the case of $T = 3$, $c \in \circledast_{t=1}^{3} \mathsf{Pos}(V_t, W_t)$ is a comb if and only if there exists $c_2 \in \mathsf{Chn}(V_1, W_1) \circledast \mathsf{Chn}(V_2, W_2)$ satisfying



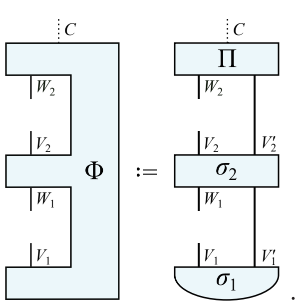
c_1 and c_2 are uniquely determined by c. The discussion can be easily extended to the case of $T \geq 4$.

D. Quantum testers

A quantum comb $\Phi \in \circledast_{t=0}^{T} \mathsf{Chn}(W_t, V_{t+1})$ with $W_0 := \mathbf{C}$ and $V_{T+1} := C$ is called a tester if it is expressed in the form

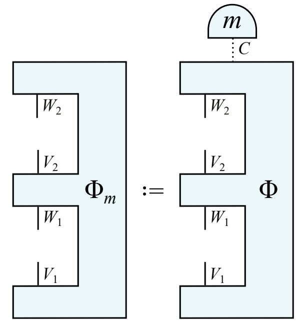
$$\Phi := \Pi \circledast \sigma_T \circledast \sigma_{T-1} \circledast \cdots \circledast \sigma_1, \tag{3}$$

where, for each $t \in \{1, \dots, T\}$, $\sigma_t \in \mathsf{Chn}(W_{t-1} \otimes V'_{t-1}, V_t \otimes V'_t)$ is a channel, $\Pi \in \mathsf{Chn}(W_T \otimes V'_T, C)$ is a measurement, and $W_0 = V'_0 = \mathbf{C}$. Let $\mathsf{Tester}(V_T, W_T; \dots; V_1, W_1)$ (or simply Tester) be the set of all testers expressed by Eq. (3). In the case of $T = 2$, Φ of Eq. (3) can be depicted by



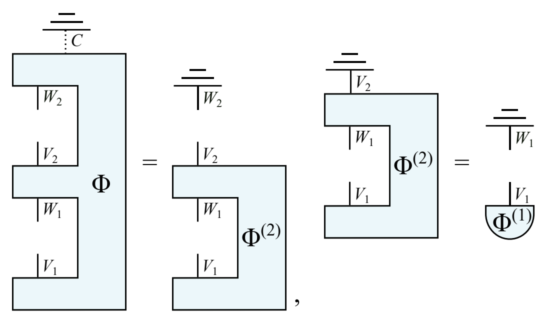
(4)

For each tester Φ and $m \in \{1, \dots, M\}$, the process Φ_m defined as



(5)

is called a tester element of Φ. A tester Φ is uniquely determined by a set of its tester elements $\{\Phi_m\}_{m=1}^{M}$. A tester can be regarded as an extension of a POVM (recall that each POVM Π is uniquely determined by a set of POVM elements $\{\Pi_m\}_{m=1}^{M}$). We assume, without loss of generality, that the classical system C satisfies $N_C = M$. Since a tester Φ is a comb, there exists a normalized state $\Phi^{(1)} \in \mathsf{Den}_{V_1}$ and a process $\Phi^{(2)} \in \mathsf{Pos}(W_1, V_2) \circledast \mathsf{Pos}(\mathbf{C}, V_1)$ satisfying



The second equation means that $\Phi^{(2)}$ is also a comb.

E. Choi-Jamiołkowski representation

Quantum processes and testers can be conveniently mathematically expressed in the so-called Choi-Jamiołkowski representations. We here give a brief overview of it.

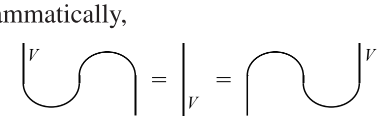
Let us consider the pure state $\cup_V \in \mathsf{Pos}_{V \otimes V}$ and the pure effect $\cap_V \in \mathsf{Pos}_{V \otimes V}$ both of which are expressed by $|\Psi\rangle_V \langle\Psi|_V$[2], where

$$|\Psi\rangle_V := \sum_{i=1}^{N_V} |i\rangle_V \otimes |i\rangle_V \in V \otimes V$$

and $\{|i\rangle_V\}_{i=1}^{N_V}$ is the standard basis of system V. One can easily verify

$$(\mathbb{1}_V \otimes \cap_V) \circ (\cup_V \otimes \mathbb{1}_V) = \mathbb{1}_V = (\cap_V \otimes \mathbb{1}_V) \circ (\mathbb{1}_V \otimes \cup_V),$$

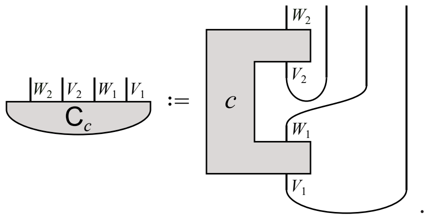
or, diagrammatically,



(6)

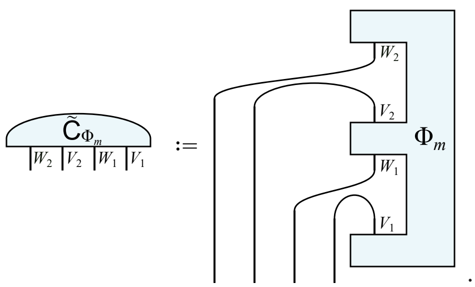
[2] $\cap_V$ can also be represented by the linear map $\mathsf{Pos}_{V \otimes V} \ni \rho \mapsto \mathrm{Tr}(\rho \cdot |\Psi\rangle\langle\Psi|) = \langle\Psi|\rho|\Psi\rangle \in \mathbf{R}_+$.

Using $\cup_V$ (or $\cap_V$), any process can be expressed by a state (or an effect). Let us consider the map C that sends a process $c \in \mathsf{Pos}(V_1, W_1) \circledast \mathsf{Pos}(V_2, W_2)$ expressed by Eq. (1) to the state $\mathsf{C}_c \in \mathsf{Pos}_{W_2 \otimes V_2 \otimes W_1 \otimes V_1}$ defined by



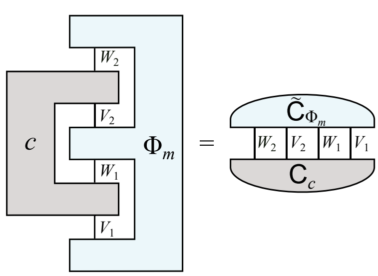
. (7)

C is an isomorphism from $\mathsf{Pos}(V_1, W_1) \circledast \mathsf{Pos}(V_2, W_2)$ to $\mathsf{Pos}_{W_2 \otimes V_2 \otimes W_1 \otimes V_1}$, or more generally from $\circledast_{t=1}^{T} \mathsf{Pos}(V_t, W_t)$ to $\mathsf{Pos}_{W_T \otimes V_T \otimes \cdots \otimes W_1 \otimes V_1}$. Let us also consider the map $\tilde{\mathsf{C}}$ that sends a tester element Φ_m expressed by Eq. (5) to the effect $\tilde{\mathsf{C}}_{\Phi_m} \in \mathsf{Pos}_{W_2 \otimes V_2 \otimes W_1 \otimes V_1}$ defined by



. (8)

$\tilde{\mathsf{C}}$ is also an isomorphism. It is easily seen from Eq. (6) that



holds. The right-hand side can be mathematically expressed by $\mathrm{Tr}(\tilde{\mathsf{C}}_{\Phi_m} \mathsf{C}_c)$. These maps are useful tools to mathematically represent a process as a positive semidefinite matrix.

It follows from Eq. (7) that a process $c \in \circledast_{t=1}^{T} \mathsf{Pos}(V_t, W_t)$ is a comb [i.e., $c \in \circledast_{t=1}^{T} \mathsf{Chn}(V_t, W_t)$] if and only if there exists a comb $c_{T-1} \in \circledast_{t=1}^{T-1} \mathsf{Chn}(V_t, W_t)$ such that

$$\mathrm{Tr}_{W_T} \mathsf{C}_c = I_{V_T} \otimes \mathsf{C}_{c_{T-1}}. \tag{9}$$

By recursively applying Eq. (9), we see that $c \in \circledast_{t=1}^{T} \mathsf{Pos}(V_t, W_t)$ is a comb if and only if there exists $\{c_t \in \circledast_{t'=1}^{t} \mathsf{Pos}(V_{t'}, W_{t'})\}_{t=1}^{T-1}$ such that

$$\begin{aligned} &\mathrm{Tr}_{W_1} \mathsf{C}_{c_1} = I_{V_1}, \\ &\mathrm{Tr}_{W_t} \mathsf{C}_{c_t} = I_{V_t} \otimes \mathsf{C}_{c_{t-1}}, \qquad \forall t \in \{2, \dots, T-1\}, \\ &\mathrm{Tr}_{W_T} \mathsf{C}_c = I_{V_T} \otimes \mathsf{C}_{c_{T-1}}. \end{aligned}$$

Similarly, it follows from Eq. (8) that a process $\Phi \in \circledast_{t=0}^{T} \mathsf{Pos}(W_t, V_{t+1})$ with $W_0 := \mathbf{C}$ and $V_{T+1} := C$ is a tester (i.e., $\Phi \in \mathsf{Tester}$) if and only if there exists a comb $\Phi' \in \circledast_{t=0}^{T-1} \mathsf{Chn}(W_t, V_{t+1})$ such that

$$\sum_{m=1}^{M} \tilde{\mathsf{C}}_{\Phi_m} = I_C \otimes \tilde{\mathsf{C}}_{\Phi'},$$

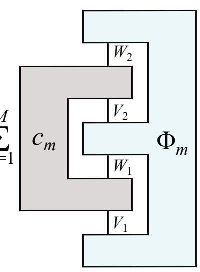
where Φ_m is defined by Eq. (5).

III. Process discrimination problems

A. Formulation

Let us consider the problem of discriminating M quantum processes $c_1, \dots, c_M \in \circledast_{t=1}^{T} \mathsf{Pos}(V_t, W_t)$ using a quantum tester. To simplify the discussion, assume that we want to maximize the cost function given by

$$P(\Phi) := \sum_{m=1}^{M}$$

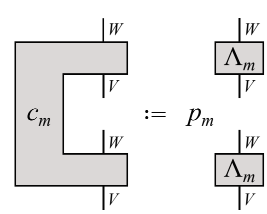


, (10)

where Φ is a tester. The problem is formulated by the following optimization problem:

$$\begin{array}{ll} \text{maximize} & P(\Phi) \\ \text{subject to} & \Phi \in \mathsf{Tester}. \end{array} \tag{P}$$

A simple example of this problem is to find a tester that maximizes the average success probability of discriminating M quantum channels $\Lambda_1, \dots, \Lambda_M \in \mathsf{Chn}(V, W)$. Let $p_m \in \mathbf{R}_+$ be the prior probability of the channel Λ_m. In the case in which two evaluations are made, the average success probability of Φ is expressed by Eq. (10) with $V_t := V$, $W_t := W$, and



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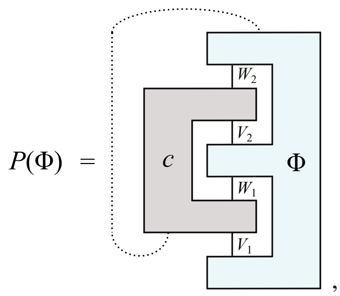
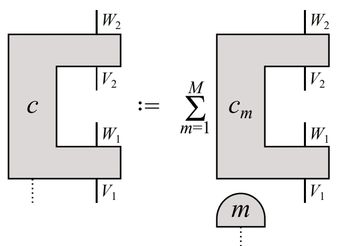
Similarly, if T evaluations are allowed, then the average success probability is expressed as in Eq. (10) with $c_m := p_m \Lambda_m^{\circledast T} \in \circledast_{t=1}^{T} \mathsf{Pos}(V, W)$. Note that there exist at least several important process discrimination problems that cannot be expressed in the form of Problem (P). To overcome this limitation, one can consider a more general setting (see [34]), in which case the diagrammatic representation described in this paper is also useful.

It is worth noting that $P(\Phi)$ can also be expressed by [39]

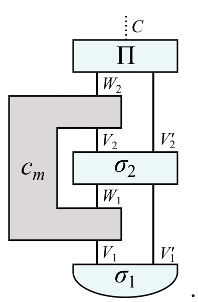


where



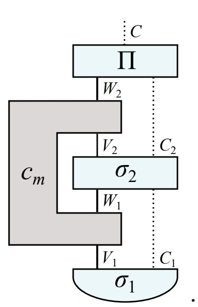
B. Special types of testers

Any tester Φ allowed by quantum theory can be expressed as in Eq. (4). Using this expression, we can depict the connection of c_m and Φ as

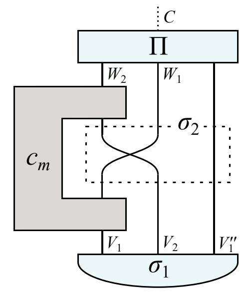


(11)

For a better understanding, let us consider two special types of quantum testers. The first type is discrimination without entanglement between the input and ancillary systems, which can be diagrammatically represented in the form



This can be regarded as a special case of Eq. (11) in which two systems $C_1 := V_1'$ and $C_2 := V_2'$ are classical. This discrimination can be adaptive; indeed, the state of W_1 may be used to adaptively control the state of V_2. The second type is nonadaptive discrimination (but unlimited entanglement is available), which can be depicted in the form



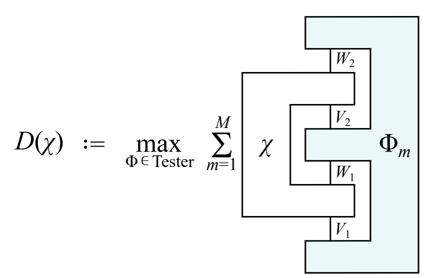
This is a special case of Eq. (11) in which $V_1' = V_2 \otimes V_1''$, $V_2' = W_1 \otimes V_1''$, and $\sigma_2 = \times_{W_1,V_2} \otimes \mathbb{1}_{V_1''}$ hold, where $\times_{V,W}$ is the process that swaps two systems V and W.

C. Dual problems

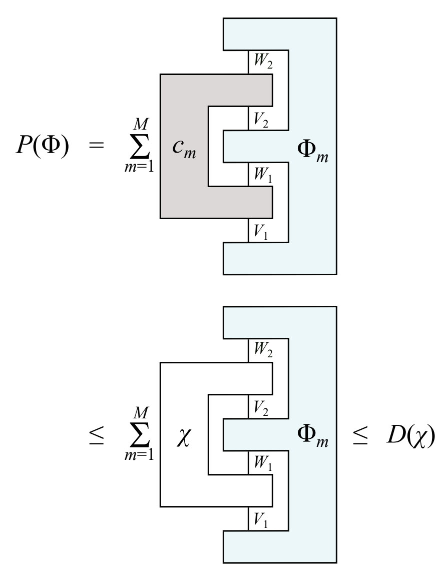
Assume that Problem (P) is given and let us consider the following optimization problem:

$$\begin{array}{ll} \text{minimize} & D(\chi) \\ \text{subject to} & \chi \in \bigotimes_{t=1}^{T} \mathsf{Pos}_{W_t \otimes V_t}, \\ & \chi \geq c_m \ (\forall m \in \{1, \dots, M\}) \end{array} \tag{DP}$$

where $\chi \geq c_m$ denotes $\mathsf{C}_\chi \geq \mathsf{C}_{c_m}$ (i.e., $\mathsf{C}_\chi - \mathsf{C}_{c_m}$ is positive semidefinite) and



Let Φ and χ be, respectively, feasible solutions to Problems (P) and (DP); then, one can easily see that



always holds. This implies that the optimal value of Problem (P) is upper bounded by that of Problem (DP). Using the Choi-Jamiołkowski representation of Φ_m (i.e., $\tilde{\mathsf{C}}_{\Phi_m}$), we can reformulate Problem (P) as a semidefinite programming problem. It follows that Problem (DP) is its dual problem and the strong duality holds (for details, see [34]). We can also derive that there exists an optimal solution to Problem (DP) that is proportional to some quantum comb. Note that Chiribella [33] derived another type of dual problem, in which the solution is restricted to be proportional to some comb. Problem (DP) can often be used to investigate the properties of optimal discrimination [34].

IV. Conclusion

We have shown a diagrammatic representation of quantum process discrimination problems, which give us an intuitive understanding of quantum processes and testers. A quantum tester can represent any kind of discrimination permitted by quantum theory, including the adaptive one. The problem of finding an optimal tester for process discrimination and its dual problem were outlined. The diagrammatic approach provides an insightful operational interpretation for process discrimination.

Acknowledgment

I am grateful to O. Hirota, K. Kato, and T. S. Usuda for support. This work was supported by JSPS KAKENHI Grant Number JP19K03658.

References

[1] M. Ban, K. Kurokawa, R. Momose, and O. Hirota, "Optimum measurements for discrimination among symmetric quantum states and parameter estimation," *Int. J. Theor. Phys.*, vol. 36, no. 6, pp. 1269–1288, 1997.

[2] T. S. Usuda, I. Takumi, M. Hata, and O. Hirota, "Minimum error detection of classical linear code sending through a quantum channel," *Phys. Lett. A*, vol. 256, pp. 104–108, 1999.

[3] S. M. Barnett, "Minimum-error discrimination between multiply symmetric states," *Phys. Rev. A*, vol. 64, no. 3, 030303, 2001.

[4] E. Andersson, S. M. Barnett, C. R. Gilson, and K. Hunter, "Minimum-error discrimination between three mirror-symmetric states," *Phys. Rev. A*, vol. 65, no. 5, 052308, 2002.

[5] M. Ježek, J. Řeháček, and J. Fiurášek, "Finding optimal strategies for minimum-error quantum-state discrimation," *Phys. Rev. A*, vol. 65, no. 6, 060301, 2002.

[6] C. L. Chou and L. Y. Hsu, "Minimum-error discrimination between symmetric mixed quantum states," *Phys. Rev. A*, vol. 68, no. 4, 042305, 2003.

[7] Y. C. Eldar and G. D. Forney Jr., "On quantum detection and the square-root measurement," *IEEE Trans. Inf. Theory*, vol. 47, no. 3, pp. 858–872, 2001.

[8] K. Kato and O. Hirota, "Square-root measurement for quantum symmetric mixed state signals," *IEEE Trans. Inf. Theory*, vol. 49, no. 12, pp. 3312–3317, 2003.

[9] Y. C. Eldar, A. Megretski, and G. C. Verghese, "Optimal detection of symmetric mixed quantum states," *IEEE Trans. Inf. Theory*, vol. 50, no. 6, pp. 1198–1207, 2004.

[10] T. Sawada, T. Tsuchimoto, and T. S. Usuda, "Quantum detection of artificial mixed-state signals," pp. 405–408, 2007.

[11] G. Cariolaro and A. Vigato, "Helstrom's theory on quantum binary decision revisited," in *Information Theory Workshop (ITW), 2011 IEEE*, pp. 242–246, IEEE, 2011.

[12] J. Tyson, "Two-sided bounds on minimum-error quantum measurement, on the reversibility of quantum dynamics, and on maximum overlap using directional iterates," *J. Math. Phys.*, vol. 51, no. 9, 092204, 2010.

[13] J. Bae, "Structure of minimum-error quantum state discrimination," *New J. Phys.*, vol. 15, no. 7, 073037, 2013.

[14] K. Nakahira and T. S. Usuda, "Quantum measurement for a group-covariant state set," *Phys. Rev. A*, vol. 87, no. 1, 012308, 2013.

[15] C. W. Helstrom, "Quantum detection and estimation theory," *J. Stat. Phys.*, vol. 1, no. 2, pp. 231–252, 1969.

[16] A. S. Holevo, "Statistical decision theory for quantum systems," *J. Multivar. Anal.*, vol. 3, pp. 337–394, 1973.

[17] H. P. Yuen, K. S. Kennedy, and M. Lax, "Optimum testing of multiple hypotheses in quantum detection theory," *IEEE Trans. Inf. Theory*, vol. 21, no. 2, pp. 125–134, 1975.

[18] M. F. Sacchi, "Optimal discrimination of quantum operations," *Phys. Rev. A*, vol. 71, no. 6, 062340, 2005.

[19] M. F. Sacchi, "Entanglement can enhance the distinguishability of entanglement-breaking channels," *Phys. Rev. A*, vol. 72, no. 1, 014305, 2005.

[20] A. Gilchrist, N. K. Langford, and M. A. Nielsen, "Distance measures to compare real and ideal quantum processes," *Phys. Rev. A*, vol. 71, no. 6, 062310, 2005.

[21] G. Wang and M. Ying, "Unambiguous discrimination among quantum operations," *Phys. Rev. A*, vol. 73, no. 4, 042301, 2006.

[22] G. Gutoski and J. Watrous, "Toward a general theory of quantum games," in *Proceedings of the 39th annual ACM symposium on Theory of computing*, pp. 565–574, 2007.

[23] L. Li and D. Qiu, "Optimal discrimination between quantum operations," *J. Phys. A: Math. Theor.*, vol. 41, no. 33, 335302, 2008.

[24] J. Watrous, "Semidefinite programs for completely bounded norms," *Theory OF Computing*, vol. 5, pp. 217–238, 2009.

[25] W. Matthews, M. Piani, and J. Watrous, "Entanglement in channel discrimination with restricted measurements," *Phys. Rev. A*, vol. 82, no. 3, 032302, 2010.

[26] M. Ziman and M. Sedlák, "Single-shot discrimination of quantum unitary processes," *J. Mod. Opt.*, vol. 57, no. 3, pp. 253–259, 2010.

[27] M. Sedlák and M. Ziman, "Optimal single-shot strategies for discrimination of quantum measurements," *Phys. Rev. A*, vol. 90, no. 5, 052312, 2014.

[28] S. Pirandola and C. Lupo, "Ultimate precision of adaptive noise estimation," *Phys. Rev. Lett.*, vol. 118, no. 10, 100502, 2017.

[29] J. ur Rehman, A. Farooq, Y. Jeong, and H. Shin, "Quantum channel discrimination without entanglement," *Quantum Inf. Process.*, vol. 17, no. 10, 271, 2018.

[30] S. Pirandola, R. Laurenza, C. Lupo, and J. L. Pereira, "Fundamental limits to quantum channel discrimination," *npj Quantum Inf.*, vol. 5, no. 1, pp. 1–8, 2019.

[31] G. Chiribella, G. M. D'Ariano, and P. Perinotti, "Quantum circuit architecture," *Phys. Rev. Lett.*, vol. 101, no. 6, 060401, 2008.

[32] G. Chiribella, G. M. D'Ariano, and P. Perinotti, "Memory effects in quantum channel discrimination," *Phys. Rev. Lett.*, vol. 101, no. 18, 180501, 2008.

[33] G. Chiribella, "Optimal networks for quantum metrology: semidefinite programs and product rules," *New J. Phys.*, vol. 14, no. 12, 125008, 2012.

[34] K. Nakahira and K. Kato, "Generalized quantum process discrimination problems," to appear.

[35] B. Coecke, "The logic of entanglement: an invitation," *Technical Report RR-03-12, Department of Computer Science, Oxford University*, 2003.

[36] S. Abramsky and B. Coecke, "A categorical semantics of quantum protocols," in *Proceedings of the 19th Annual IEEE Symposium on Logic in Computer Science*, pp. 415–425, IEEE, 2004.

[37] B. Coecke and A. Kissinger, *Picturing quantum processes*. Cambridge University Press, 2017.

[38] K. Nakahira, "Derivation of quantum theory with superselection rules," *Physical Review A*, vol. 101, no. 2, 022104, 2020.

[39] K. Nakahira, "Discrimination of symmetric states in operational probabilistic theory," *Tamagawa University Quantum ICT Research Institute Bulletin*, vol. 9, no. 1, pp. 15–27, 2019.