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***Abstract*—We compute the random coding bound for a continuous classical-quantum channel with unsqueezed Gaussian states numerically. In addition we compare it with that of discrete classical-quantum channel with PSK coherent states under the same energy constraint.**

I. Introduction

The reliability function performs an important role in the classical-quantum channel coding theorem. It shows the speed of the exponential decay of the error probability at rates R below the capacity as the code length goes to infinity. The reliability function of the discrete classical-quantum channel was studied by Holevo and Burnashev [1]. On the analogy from the classical case, they defined the random coding bound and the expurgated bound based on quantum channel coding and proved that these give lower bounds for the reliability function truly in the pure state case [1]. Then Holevo proved the expurgated bound also holds in the mixed state case [5], while the random coding bound for mixed states is yet to be proved. Moreover, he extended these results to a continuous channel with constrained inputs.

The computation of lower bounds for the reliability function is itself a quite interesting and nontrivial problem, which involves optimization with respect to an *a priori* probability distribution and so on. Kato derived the optimal distributions of the random coding bound, the expurgated bound and the zero-rate reliability function for M-ary PSK coherent state signals analytically [6]. On the other hand, lower bounds of the reliability function of continuous classical-quantum channels with Gaussian states were studied by Holevo, Sohma and Hirota. Holevo calculated the expurgated bound for coherent states [3]. Holevo, Sohma and Hirota calculated the expurgated bound for coherent states with thermal noise [5]. The random coding bound cannot be computed analytically even for coherent states.

In this paper we calculate the random coding bound of continuous classical-quantum channel for coherent states with thermal noise numerically. Although the random coding bound for mixed states has not been proved, we assume Holevo's conjecture [5] holds and compute it. If we consider only pure state signals, our results are valid strictly. Furthermore, we compare the random coding bound of the discrete classical-quantum channel with M-ary PSK coherent states with that of the continuous classical-quantum channel with coherent states under the same energy constraint.

II. Quantum Reliability Function

Let $\mathcal{A} = \{1, 2, \ldots, M\}$ be an input alphabet and $\mathcal{B} = \{\rho_1, \ldots, \rho_M\}$ a set of quantum states, which are described by density operators in a Hilbert space $\mathcal{H}$. Then a discrete classical-quantum channel is characterized by a map $\Theta : \mathcal{A} \ni k \to \rho_k \in \mathcal{B}$. We consider a codebook W defined by

$$W = \{w_j = (a_j^{(1)}, \ldots, a_j^{(n)}); j = 1, \ldots, M'\} \quad (1)$$

and a decoding process Π represented by a positive operator valued measure(POVM)

$$\Pi = \left\{ \Pi_j; \Pi_j \geq 0, \ \sum_j \Pi_j = I \right\}, \quad (2)$$

where I is the identity operator on the n-th tensor of the signal Hilbert space $\mathcal{H}^{\otimes n}$. Here the codeword w_j corresponds to a quantum state

$$\Theta[w_j] = \Theta[a_j^{(1)}] \otimes \cdots \otimes \Theta[a_j^{(n)}], \quad (3)$$

through the classical-quantum channel Θ. Then the minimum probability of decoding error is given by

$$P(n, M') = \min_{W,\Pi} \frac{1}{M'} \sum_{j=1}^{M'} (1 - \mathrm{Tr}\Theta[w_j]\Pi_j). \quad (4)$$

When the communication rate is fixed to R [nats/letter], there is a relationship between the code length n and the number of messages M' such that $M' = e^{nR}$. Using the minimum probability of decoding error (4), we introduce the quantum reliability function

$$E^d(R) = \limsup_{n\to\infty} \frac{1}{n} \log \frac{1}{P(n, e^{nR})}. \quad (5)$$

The quantum reliability function should be estimated by upper and lower bounds because it is difficult to compute it directly. One of lower bounds is known as the random coding bound. It is expected that the inequality

$$P(n, e^{nR}) \leq 2\exp[-n(\mu(\pi, s) - sR)] \tag{6}$$

holds for $0 \leq s \leq 1$ [2], where

$$\mu(\pi, s) = -\log \mathrm{Tr}\left[\sum_{k=1}^{M} \pi_k \rho_k^{\frac{1}{1+s}}\right]^{1+s}, \tag{7}$$

and $\pi = (\pi_1, \cdots, \pi_M)$ is an *a priori* probability distribution on the input alphabet $\mathcal{A}$. Note that the inequality (6) has been proved only in the pure state case [2]. The random coding bound of a discrete classical-quantum channel is defined by

$$E_r^d(R) = \max_{0\leq s\leq 1} \max_{\pi}[\mu(\pi, s) - sR] \tag{8}$$

for $0 < R < C$. Here C is the channel capacity given by

$$C = \sup_{\pi}\left[H\left(\sum_{k=1}^{M} \pi_k\rho_k\right) - \sum_{k=1}^{M} \pi_k H(\rho_k)\right], \tag{9}$$

where $H(\rho) \equiv -\mathrm{Tr}\rho\log\rho$ is the von Neumann entropy for a density operator ρ. Then using the inequality (6) we have

$$E^d(R) \geq E_r^d(R). \tag{10}$$

In this paper we consider a continuous classical-quantum channel and consider its quantum reliability function. We take as the input alphabet a finite dimensional Euclidean space $\mathbb{R}^d$. Then the continuous classical-quantum channel is described by a weakly continuous map $\Theta : \mathbb{R}^d \ni x \to \rho_x \in \mathfrak{S}(\mathcal{H})$, where $\mathfrak{S}(\mathcal{H})$ is the set of all quantum states. We consider a codebook W_{N_s} defined by

$$W_{N_s} = \{w_j = (x_j^{(1)}, \ldots, x_j^{(n)})\ ; j = 1, \ldots, M'\}, \tag{11}$$

where we impose an energy constraint on the codeword w_j as

$$\sum_{i=1}^{n} \mathcal{E}(x_j^{(i)}) \leq nN_s, \tag{12}$$

with a predetermined energy function $\mathcal{E}$. The decoding process Π represented by the POVM (2). The codeword w_j corresponds to a quantum state

$$\Theta[w_j] = \Theta[x_j^{(1)}] \otimes \cdots \otimes \Theta[x_j^{(n)}] \tag{13}$$

through the continuous classical-quantum channel Θ. Then the minimum probability of decoding error is given by

$$P(n, M') = \min_{W_{N_s},\Pi} \frac{1}{M'}\sum_{j=1}^{M'}(1 - \mathrm{Tr}\Theta[w_j]\Pi_j^{(n)}). \tag{14}$$

The random coding bound for the continuous classical-quantum channel is defined as follows [2],

$$E_r(R) = \max_{0\leq s\leq 1} \max_{0\leq p} \max_{\pi\in\mathcal{P}_1}(\mu(\pi, s, p) - sR) \tag{15}$$

for $0 < R < C$. Here

$$\mu(\pi, s, p) = -\log \mathrm{Tr}\left[\int e^{\frac{p}{2}[\mathcal{E}(x)-N_s]}\rho_x^{\frac{1}{1+s}}\pi(dx)\right]^{1+s} \tag{16}$$

is the quantum Gallager function and $\mathcal{P}_1$ is the set of probability distributions π that satisfy

$$\int \mathcal{E}(x)\pi(dx) < N_s, \tag{17}$$

and the channel capacity C is given by

$$C = \sup_{\pi\in\mathcal{P}_1}\left[H\left(\int \rho_x\pi(dx)\right) - \int H(\rho_x)\pi(dx)\right], \tag{18}$$

As in the discrete case, it is proved that

$$E(R) \geq E_r(R) \tag{19}$$

for pure states.

III. Lower Bounds of Quantum Reliability Function For Gaussian States

As a continuous classical-quantum channel, we consider a classical-quantum Gaussian channel, which is defined by a channel map $\Theta : \mathbb{R}^2 \ni m \to \rho_m \in \mathfrak{S}(\mathcal{H})$, where ρ_m is single-mode quantum Gaussian state given by mean $m = \begin{pmatrix} m_1 \\ m_2 \end{pmatrix}$ and the fixed correlation matrix $\alpha = \begin{pmatrix} \alpha_{11} & \alpha_{12} \\ \alpha_{12} & \alpha_{22} \end{pmatrix}$ [3]. The energy function of the quantum Gaussian state is given by

$$\mathcal{E}(m) = \frac{1}{2}\left(m_1^2 + m_2^2\right). \tag{20}$$

In the following we restrict ourselves to an unsqueezed Gaussian state, which has a correlation matrix given by

$$\begin{pmatrix} \lambda\hbar & 0 \\ 0 & \lambda\hbar \end{pmatrix}, \quad \lambda = \frac{1}{2} + N, \tag{21}$$

where $\hbar$ is the Dirac constant and N corresponds to a thermal noise. It is known that the Gallager function for such quantum Gaussian states is calculated as follows

$$\begin{aligned} \mu(\pi_o, s, p) = (1+s)\log f_{\frac{1}{1+s}}(\lambda) + p(1+s)N_s/2 \\ + \log[A(s,p)^{1+s} - B(s,p)^{1+s}], \end{aligned} \tag{22}$$

where π_o represents the optimal *a priori* probability distribution which is assumed to be a normal distribution with the correlation matrix $\begin{pmatrix} N_s\hbar & 0 \\ 0 & N_s\hbar \end{pmatrix}$ and the mean 0. The functions included in Eq. (22) are given by

$$\begin{aligned} f_s(t) &= (t+1/2)^s - (t-1/2)^s \\ g_s(t) &= \tfrac{1}{2t}\tfrac{(t+1/2)^s+(t-1/2)^s}{(t+1/2)^s-(t-1/2)^s} \\ A(s,p) &= \left(\lambda g_{\frac{1}{1+s}}(\lambda) + 1/2\right)(1 - pN_s/2) + N_s \\ B(s,p) &= \left(\lambda g_{\frac{1}{1+s}}(\lambda) - 1/2\right)(1 - pN_s/2) + N_s \end{aligned}$$

The expression (22) holds for $A(s,p) > B(s,p)$ or

$$0 \leq p \leq \frac{2}{N_s}. \tag{23}$$

When the communication rate R is close to the channel capacity C, the random coding bound is closer to the reliability function than the expurgated bound. It is known that when the communication rate is

$$R \leq \frac{\partial}{\partial s}\mu(\pi_o, 1, p(N_s)), \tag{24}$$

where

$$p(N_s) = \frac{1}{g} + \frac{2}{N_s} - \frac{2\vartheta(N_s/g)}{N_s}, \tag{25}$$

with $\vartheta(t) = \frac{1+\sqrt{t^2+1}}{2}$ and $g = \lambda g_{\frac{1}{2}}(\lambda)$, the random coding bound is a straight line with a slope of -1 [5]. Then, $E_r(R)$ is given by

$$E_r(R) = \mu(\pi_o, 1, p(N_s)) - R. \tag{26}$$

Note that the channel capacity is given by

$$\begin{aligned} C =&(N_s + N + 1)\log(N_s + N + 1) \\ &- (N_s + N)\log(N_s + N) \\ &- (N + 1)\log(N + 1) + N\log(N). \end{aligned} \tag{27}$$

Let us calculate the random coding bound by optimizing s and p of $\mu(\pi_o, s, p) - sR$. We adopt a very primitive optimization technique where we discretize $0 \leq s \leq 1$ and $0 \leq p < \frac{2}{N_s}$ in $\Delta s = 10^{-3}$ and $\Delta p = 10^{-3}$ increments respectively, and find the maximum value. Finding the maximum value of $\mu(\pi_o, s, p)$ with respect to p, we obtain

$$m(s) = \max_{0 \leq p < \frac{2}{N_s}} \mu(\pi_o, s, p) - sR. \tag{28}$$

Fig. 1 shows graph of $m(s)$ for the average photon number of signal state $N_s = 0.5$ and the thermal noise $N = 0.1$, where $R = 0.723$ [nats/letter] is the value of capacity C.

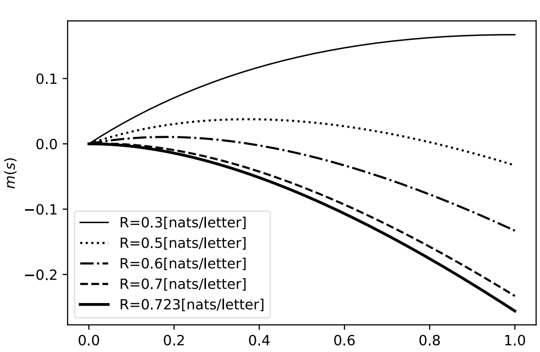
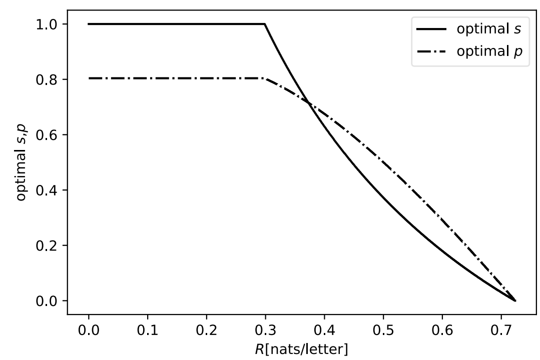
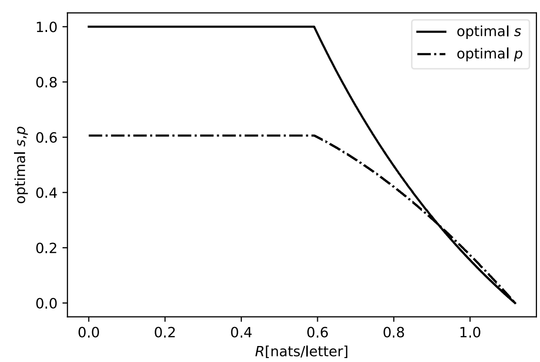


Fig. 1. $m(s)$ for $N_s = 0.5, N = 0.1$.

Maximizing $m(s)$ with respect to s, we obtain the random coding bound. Fig. 2 shows the optimal values of s and p for a given communication rate R. Fig. 2-(a) shows the optimal values of s and p for $N_s = 0.5$ and $N = 0.1$. Fig. 2-(b) shows the optimal values of s and p for $N_s = 1$ and $N = 0.1$. Fig. 3 shows graphs of the random coding bound. Here we consider two subparameters, N_s and N. Fig. 3-(a) shows graphs of the random coding bound for the average photon number of signal state $N_s = 0.5, 1, 2$ and the thermal noise $N = 0.1$. Fig. 3-(b) shows graphs of the random coding bound for the average photon number of signal state $N_s = 0.5$ and the thermal noise $N = 0.1, 0.2, 0.5$.



(a)



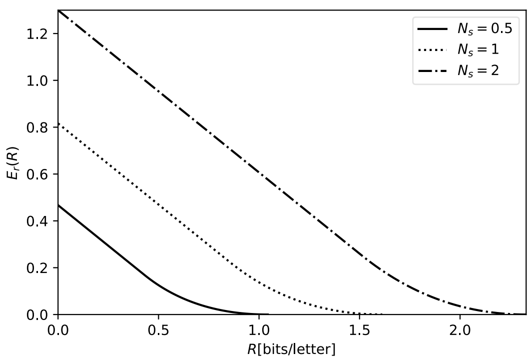
(b)

Fig. 2. (a) The optimal values of s and p for the random coding bound in the case of $N_s = 0.5$, $N = 0.1$; (b) The optimal values of s and p for the random coding bound in the case of $N_s = 1$, $N = 0.1$.

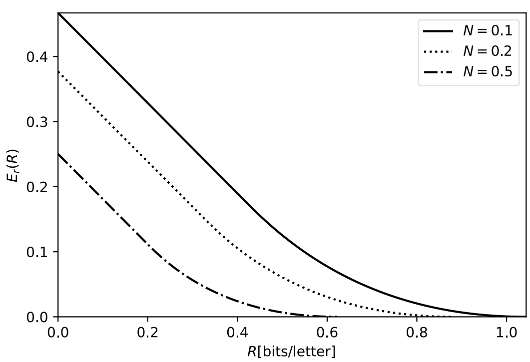
We compare the random coding bound $E_r(R)$ for the continuous channel with the random coding bound $E_r^d(R)$ for the discrete channel. In the discrete channel we use M-ary PSK coherent state signals [6],

$$\mathcal{B} = \left\{ |\psi_k\rangle = |\alpha \exp[\mathbf{i}\frac{2\pi(k-1)}{M}]\rangle; k = 1, ..., M \right\}. \tag{29}$$

Every signal $|\psi_k\rangle$ in $\mathcal{B}$ has the same amplitude $|\alpha|^2$, and the average photon number of these signals is given by $|\alpha|^2$, which is independent of any *a priori* distribution. So we consider the continuous classical-quantum channel with coherent states of the average photon number $N_s = |\alpha|^2$.

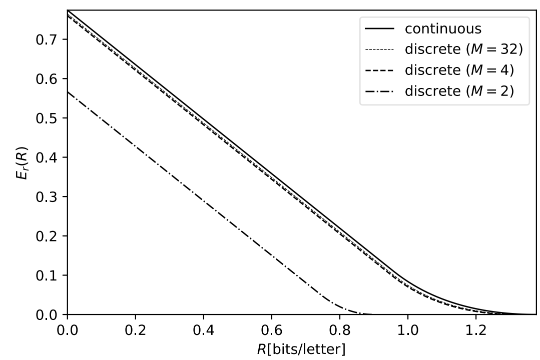


(a)

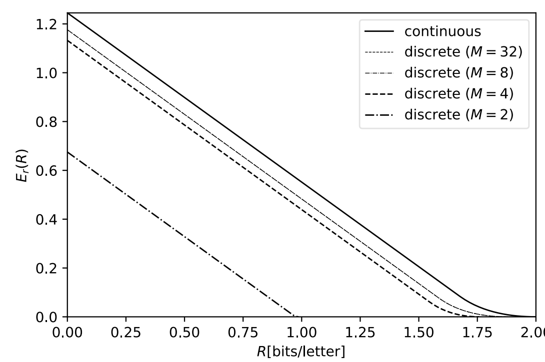


(b)

Fig. 3. The random coding bounds for continuous classical-quantum channels with unsqueezed state signals:(a)$N = 0.1$,$N_s = 0.5, 1, 2$; (b)$N_s = 0.5$, $N = 0.1, 0.2, 0.5$.



(a)



(b)

Fig. 4. (a) Comparison of the random coding bounds for continuous and discrete channels in the case of $N_s = 0.5$, $N = 0$: (b) Comparison of the random coding bounds for continuous and discrete channels in the case of $N_s = 1$, $N = 0$.

Fig. 4 shows the random coding bounds $E_r(R)$ for the continuous channel and the random coding bound $E_r^d(R)$ for M-ary PSK states, where the average photon number is $N_s = 0.5, 1$. Fig. 4-(a) compares $E_r(R)$ and $E_r^d(R)$ at $M = 2, 4, 32$. In this case, the random coding bound $E_r^d(R)$ for $M = 4$ and $M = 32$ are nearly equal. Fig. 4-(b) compares $E_r(R)$ and $E_r^d(R)$ at $M = 2, 4, 8, 32$. In this case, the random coding bound $E_r^d(R)$ for $M = 8$ and $M = 32$ are nearly equal. When $N_s = 0.5$, $E_r^d(R)$ is nearly equal to $E_r(R)$ at $M = 8$. On the other hand, when $N_s = 1$, the random coding bound $E_r^d(R)$ does not achieve $E_r(R)$. Therefore we need to find another signal states other than M-ary PSK states to achieve the random coding bound of the continuous channel.

IV. Conclusion

We have computed the random coding bound $E_r(R)$ for a continuous classical-quantum channel with unsqueezed Gaussian states. We have compared it with the random coding bound $E_r^d(R)$ of the discrete channel with M-ary PSK coherent states. We will find another signal set $\mathcal{B}$ of the discrete classical-quantum channel which has the random coding bound achieving $E(R)$ asymptotically when $N_s > 1$, in the future work.

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