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Kenji Nakahira

Quantum Information Science Research Center,
Quantum ICT Research Institute, Tamagawa University
6-1-1 Tamagawa-gakuen, Machida, Tokyo 194-8610 Japan

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Kenji Nakahira
Quantum Information Science Research Center,
Quantum ICT Research Institute, Tamagawa University
6-1-1 Tamagawa-gakuen, Machida, Tokyo 194-8610 Japan
E-mail: nakahira@lab.tamagawa.ac.jp

***Abstract*—Classical wavelet transforms have been successfully applied in many fields of signal processing. Quantum wavelet transforms, which are the quantum analogues of the classical wavelet transforms, are expected to be a promising tool for quantum information and quantum computation. We propose an efficient implementation of any quantum (decimated) orthogonal wavelet transform and its undecimated version.**

I. Introduction

Quantum Fourier transforms have been extensively used in the field of quantum computing. In the signal processing community, classical wavelet transforms have been often used instead of classical Fourier transforms [1], [2]. This implies that quantum wavelet transforms have a great potential for quantum signal processing and quantum computing. Although efficient implementations of quantum orthogonal wavelet transforms (QOWTs) have been proposed only for the Haar and Daubechies filters [3]–[6], there are other commonly used orthogonal wavelet filters, such as Symlet [7] and Coiflet [8]. To get good performance for required tasks, the users need to choose an appropriate wavelet filter. Thus, it is natural to find a way of implementing QOWTs to cope with these filters.

In this paper, we derive a new factorization of the wavelet transform matrix for any QOWT, which leads to an efficient quantum circuit. As well as the OWTs, those undecimated versions have also been widely used in classical signal processing, such as pattern recognition and denoising, due to the advantage of the shift-invariant property. We show that our approach can be easily extended to the implementation for the undecimated version of the OWT (UWT).

II. Wavelet transforms (WTs)

A. Preliminaries

Let $\mathcal{U}_m$ be the set of all unitary matrices of order m. Matrices are written in upper-case letters, while vectors are written in bold lower-case letters. The $(j+1, k+1)$-th element of a matrix M is denoted by $M_{j,k}$. Also, the $(j+1)$-th element of a vector $\boldsymbol{v}$ is denoted by v_j. Note that the indices j and k start from zero. Let $\mathcal{I}_k := \{0, 1, \dots, k-1\}$. O_m and $\mathbf{0}_m$, respectively, denote the $m \times m$ zero matrix and the m-dimensional zero vector. In this paper, we consider a complex-valued input signal with $N := 2^n$ samples. When we write $k \in \mathcal{I}_N$ in binary notation as $k = q_{n-1}q_{n-2}\cdots q_0 \quad (q_0, \dots, q_{n-1} \in \mathcal{I}_2)$, which satisfies

$$k = \sum_{t=0}^{n-1} q_t 2^t,$$

the corresponding element $|k\rangle$ of the computational basis $\{|k\rangle : k \in \mathcal{I}_N\}$ is given by the tensor products of the form

$$|k\rangle = |q_{n-1}\rangle\, |q_{n-2}\rangle \cdots |q_0\rangle\,.$$

B. Classical wavelet transforms

Before passing to the main topic, let us briefly review classical orthogonal wavelet transforms (OWTs) (for details see [1], [2] and references therein). Let Ξ be the number of levels in the transform. Let $N_\xi := 2^{-\xi}N$ for each level, $\xi \in \mathcal{I}_\Xi$, of decomposition. In the OWT, as the level increases, more and more detailed information is removed. Given an input vector $\boldsymbol{x} \in \mathbb{C}^N$, the OWT calculates the scaling coefficients, $\boldsymbol{s}^{(\xi+1)} \in \mathbb{C}^{N_{\xi+1}}$, and the wavelet coefficients, $\boldsymbol{w}^{(\xi+1)} \in \mathbb{C}^{N_{\xi+1}}$, at different scales $\xi \in \mathcal{I}_\Xi$, using the following equation:

$$s_k^{(\xi+1)} = \sum_{t=0}^{2L-1} h_t s_{2k+t}^{(\xi)}, \quad w_k^{(\xi+1)} = \sum_{t=0}^{2L-1} g_t s_{2k+t}^{(\xi)}, \quad k \in \mathcal{I}_{N_{\xi+1}}, \tag{1}$$

where $\boldsymbol{s}^{(0)} := \boldsymbol{x}$. $\{h_t\}_{t=0}^{2L-1}$ and $\{g_t\}_{t=0}^{2L-1}$ are, respectively, the scaling and wavelet filters with length $2L$ (where $h_{2L-1} = g_{2L-1} = 0$ is allowed). The scaling and wavelet coefficients, respectively, correspond to low- and high-frequency components. By convention, let $h_t = g_t = 0$ for any $t \notin \mathcal{I}_{2L}$. These filters must satisfy the perfect reconstruction conditions:

$$\sum_{t=-\infty}^{\infty} h_t^* h_{t+2k} = \sum_{t=-\infty}^{\infty} g_t^* g_{t+2k} = \delta_{k,0},$$
$$\sum_{t=-\infty}^{\infty} h_t^* g_{t+2k} = 0, \tag{2}$$

where $\delta_{a,b}$ is the Kronecker delta. The output of the OWT is $[\boldsymbol{s}^{(\Xi)\mathsf{T}}, \boldsymbol{w}^{(\Xi)\mathsf{T}}, \dots, \boldsymbol{w}^{(1)\mathsf{T}}]^{\mathsf{T}} \in \mathbb{C}^N$, where $^{\mathsf{T}}$ is the transpose.

Given an even number M with $M \geq 2L$, let $H_M \in \mathbb{C}^{M\times M}$ be a block circulant matrix defined as

$$H_M := \begin{bmatrix} H^{(0)} & H^{(1)} & \cdots & H^{(\frac{M}{2}-1)} \\ H^{(\frac{M}{2}-1)} & H^{(0)} & \ddots & \vdots \\ \vdots & \ddots & \ddots & H^{(1)} \\ H^{(1)} & \cdots & H^{(\frac{M}{2}-1)} & H^{(0)} \end{bmatrix}, \tag{3}$$

where

$$H^{(k)} := \begin{bmatrix} h_{2k} & h_{2k+1} \\ g_{2k} & g_{2k+1} \end{bmatrix}. \tag{4}$$

Note that $H^{(k)} = O_2$ holds for each $k \geq L$. It is seen from Eq. (2) that $H_M H_M^{\dagger} = I_M$ holds (where † is the conjugate transpose and I_M is the identity matrix of order M), i.e., H_M is a unitary matrix. At each level ξ, $\boldsymbol{s}^{(\xi+1)}$ and $\boldsymbol{w}^{(\xi+1)}$ are obtained by

$$\boldsymbol{\eta}^{(\xi+1)} = H_{N_\xi} \boldsymbol{s}^{(\xi)},$$

where

$$\boldsymbol{\eta}^{(\xi+1)} := [s_0^{(\xi+1)}, w_0^{(\xi+1)}, s_1^{(\xi+1)}, w_1^{(\xi+1)}, \dots, s_{N_{\xi+1}-1}^{(\xi+1)}, w_{N_{\xi+1}-1}^{(\xi+1)}]^{\mathsf{T}}.$$

Let $\Pi_M \in \mathcal{U}_M$ be the matrix whose $(j+1, k+1)$-th element is[1]

$$(\Pi_M)_{j,k} := \delta_{j, \lfloor k/2 \rfloor + \nu(M,k)},$$
$$\nu(M,k) := \begin{cases} 0, & k \text{ is even}, \\ M/2, & k \text{ is odd}, \end{cases}$$

where $\lfloor x \rfloor$ is the greatest integer that is not greater than x. For any $\boldsymbol{x} \in \mathbb{C}^M$, Π_M satisfies

$$\Pi_M \boldsymbol{x} = [x_0, x_2, \dots, x_{M-2}, x_1, x_3, \dots, x_{M-1}]^{\mathsf{T}},$$

which can be interpreted that Π_M divides $\boldsymbol{x}$ into two disjoint sets of samples, even-indexed samples $[x_0, x_2, \dots]$ and odd-indexed samples $[x_1, x_3, \dots]$. The transform matrix $W \in \mathcal{U}_N$ of the OWT is expressed by

$$W := W_{\Xi-1} W_{\Xi-2} \cdots W_0, \tag{5}$$

where

$$W_\xi := (\Pi_{N_\xi} \oplus I_{N-N_\xi})(H_{N_\xi} \oplus I_{N-N_\xi}).$$

W is completely characterized by the set of L matrices $\{H^{(k)}\}_{k=0}^{L-1}$. We can easily verify that

$$W_\xi \boldsymbol{s}^{(\xi)} = \begin{bmatrix} \boldsymbol{s}^{(\xi+1)} \\ \boldsymbol{w}^{(\xi+1)} \end{bmatrix}$$

and

$$W\boldsymbol{x} = \begin{bmatrix} \boldsymbol{s}^{(\Xi)} \\ \boldsymbol{w}^{(\Xi)} \\ \vdots \\ \boldsymbol{w}^{(0)} \end{bmatrix}.$$

[1] Π_M corresponds $\Pi_M^{\dagger}$ in Ref. [4].

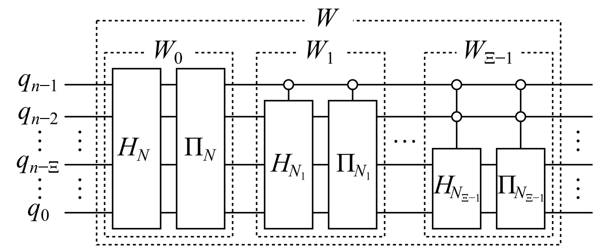

Fig. 1. Quantum circuit for the QOWT with the transform matrix W.

Assume that the order of H_{N_ξ} (i.e., N_ξ) is not less than $4L-2$ for any $\xi \in \mathcal{I}_\Xi$, which means that $N \geq 2^{\Xi}(2L-1)$ holds; we can pad $\boldsymbol{x}$ with some values (e.g., zeros) to get a sufficient number of samples.

C. Quantum wavelet transforms (QWTs)

Let us consider the following unitary transformation represented by the unitary operator

$$\hat{W} := \sum_{j=0}^{N-1} \sum_{k=0}^{N-1} W_{j,k} \left| j \right\rangle \left\langle k \right|,$$

where $W_{j,k}$ is the $(j+1, k+1)$-th element of the transform matrix W given by Eq. (5). For an input quantum state $|x\rangle$, $\hat{W}|x\rangle$ can be represented as $W\boldsymbol{x}$, where $\boldsymbol{x} \in \mathbb{C}^N$ is the column vector representation of $|x\rangle$ in the computational basis, whose $(k+1)$-th element is $x_k := \langle k|x\rangle$. For simplicity, $\hat{W}$ is identified with W.

III. Implementation of QOWTs

The QOWT with the transform matrix W can be implemented in the quantum circuit shown in Fig. 1, which operates on n input qubits. Each box with a vertical line indicates a controlled gate, where the empty circle marks the control qubit; the operation is applied if the state of each control qubit is $|0\rangle$. It is known that $\Pi_{N_\xi} \oplus I_{N-N_\xi}$ can be implemented in $O(n)$ controlled-NOT gates [4]. In this section, we show that $H_{N_\xi} \oplus I_{N-N_\xi}$ can be implemented with a complexity of $O(n)$, which indicates that W can be implemented with a complexity of $O(\Xi n)$.

Assume, without loss of generality, that h_m and g_m satisfy

$$|h_0|^2 + |g_0|^2 \neq 0, \quad H^{(L-1)} \neq O_2. \tag{6}$$

The first equation indicates that at least either h_0 or g_0 is not zero; the second equation indicates that at least either h_{2L-2}, h_{2L-1}, g_{2L-2}, or g_{2L-1} is not zero. Let $Q_M \in \mathcal{U}_M$ be the downshift permutation matrix, whose $(j+1, k+1)$-th element is

$$(Q_M)_{j,k} := \delta_{(j+1) \bmod M, k},$$

where $a \bmod b$ denotes the remainder in the division of a by b. For any $\boldsymbol{x} \in \mathbb{C}^M$, we have

$$Q_M \boldsymbol{x} = [x_1, \dots, x_{M-1}, x_0]^{\mathsf{T}}.$$

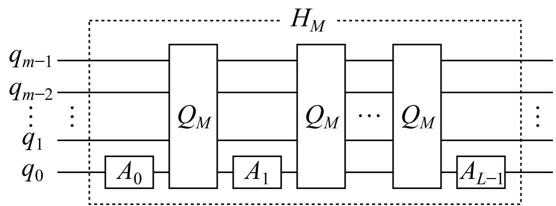

Fig. 2. Quantum circuit for H_M.

Q_M can be implemented with a complexity of $O(\log_2 M)$ [4], [9].

We show the following theorem.

Theorem 1: For any $H_M \in \mathcal{U}_M$ with $M \geq 4L-2$ expressed in the form of Eq. (3), there exists a set of L matrices $\{A_k \in \mathcal{U}_2\}_{k=0}^{L-1}$ satisfying

$$\begin{aligned} H_M = (I_{\frac{M}{2}} \otimes A_{L-1}) Q_M (I_{\frac{M}{2}} \otimes A_{L-2}) Q_M \\ \times \cdots \times Q_M (I_{\frac{M}{2}} \otimes A_0). \end{aligned} \quad (7)$$

A circuit for the implementation of H_M based on Eq. (7) is shown in Fig. 2, which has a complexity of $O(L \log_2 M)$.

To prove this theorem, we can use the following lemma (proved later).

Lemma 2: Assume $L > 1$. For any $H_M \in \mathcal{U}_M$ with $M \geq 4L-2$ expressed in the form of Eq. (3), there exist two matrices $A \in \mathcal{U}_2$ and $C \in \mathcal{U}_M$ satisfying

$$\begin{aligned} H_M &= (I_{\frac{M}{2}} \otimes A) Q_M C, \\ C_{j,k} &:= \begin{cases} c_{(k-j) \bmod M}, & j \text{ is even}, \\ d_{(k-j+1) \bmod M}, & j \text{ is odd}, \end{cases} \end{aligned} \quad (8)$$

where $c_m = d_m = 0 \ \ (\forall m \geq 2L-2)$ holds and at least either c_{2L-4} or c_{2L-3} is not zero.

The matrix C in Lemma 2 is the block circulant matrix expressed by

$$C = \begin{bmatrix} C^{(0)} & C^{(1)} & \cdots & C^{(\frac{M}{2}-1)} \\ C^{(\frac{M}{2}-1)} & C^{(0)} & \ddots & \vdots \\ \vdots & \ddots & \ddots & C^{(1)} \\ C^{(1)} & \cdots & C^{(\frac{M}{2}-1)} & C^{(0)} \end{bmatrix} \quad (9)$$

with

$$C^{(k)} := \begin{bmatrix} c_{2k} & c_{2k+1} \\ d_{2k} & d_{2k+1} \end{bmatrix},$$

where $C^{(k)} = O_2$ holds for any $k \geq L-1$.

Proof of Theorem 1 . The case $L = 1$ is obvious, since $H_M = I_{\frac{M}{2}} \otimes H^{(0)}$ holds. In the case of $L > 1$, by iteratively substituting C into H_M in Eq. (8) and applying Lemma 2, we obtain Eq. (7). ■

We now prove Lemma 2.

Proof of Lemma 2 . Let us express $A \in \mathcal{U}_2$ as

$$A = \begin{bmatrix} a_0 & a_1 \\ b_0 & b_1 \end{bmatrix}.$$

Let $D := Q_M C$. It is easily seen that D can be expressed by

$$D = \begin{bmatrix} D^{(0)} & D^{(1)} & \cdots & D^{(\frac{M}{2}-1)} \\ D^{(\frac{M}{2}-1)} & D^{(0)} & \ddots & \vdots \\ \vdots & \ddots & \ddots & D^{(1)} \\ D^{(1)} & \cdots & D^{(\frac{M}{2}-1)} & D^{(0)} \end{bmatrix},$$

where

$$D^{(k)} := \begin{bmatrix} d_{2k} & d_{2k+1} \\ c_{(2k-2) \bmod M} & c_{(2k-1) \bmod M} \end{bmatrix}.$$

From Eqs. (3) and (8), $H_M = (I_{\frac{M}{2}} \otimes A) D$ is equivalent to

$$H^{(k)} = A D^{(k)}, \quad k \in \mathcal{I}_{\frac{M}{2}}. \quad (10)$$

Since the product of unitary matrices is also unitary, one can easily see that $C = Q_M^{-1} (I_{\frac{M}{2}} \otimes A^{-1}) H_M$ is unitary if $A \in \mathcal{U}_2$ holds. Thus, it suffices to show that there exist $A \in \mathcal{U}_2$ and $\{c_k, d_k\}_{k=0}^{2L-3}$ satisfying Eq. (10).

Substituting $k = 0$ into Eq. (10) gives $H^{(0)} = A D^{(0)}$, i.e.,

$$\begin{bmatrix} a_0 & a_1 \\ b_0 & b_1 \end{bmatrix} \begin{bmatrix} d_0 & d_1 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} h_0 & h_1 \\ g_0 & g_1 \end{bmatrix}.$$

where we use $c_{M-2} = c_{M-1} = 0$, which follows from Eq. (8). Since A is unitary, d_0, d_1, a_0, and b_0 are expressed by

$$d_0 = \gamma R, \quad d_1 = s d_0, \quad a_0 = h_0 d_0^{-1}, \quad b_0 = g_0 d_0^{-1}, \quad (11)$$

where $R := \sqrt{|h_0|^2 + |g_0|^2}$ (note that $R > 0$ holds from Eq. (6)), γ is a complex number with unit modulus, and $s = h_0^{-1} h_1$ if $h_0 \neq 0$, and $s = g_0^{-1} g_1$ otherwise.

Substituting $k = L-1$ into Eq. (10) yields $H^{(L-1)} = A D^{(L-1)}$, i.e.,

$$\begin{bmatrix} a_0 & a_1 \\ b_0 & b_1 \end{bmatrix} \begin{bmatrix} 0 & 0 \\ c_{2L-4} & c_{2L-3} \end{bmatrix} = \begin{bmatrix} h_{2L-2} & h_{2L-1} \\ g_{2L-2} & g_{2L-1} \end{bmatrix}.$$

where we use $d_{M-2} = d_{M-1} = 0$, which follows from Eq. (8). Let $R' := \sqrt{|h_{2L-1}|^2 + |g_{2L-1}|^2}$ and $R'' := \sqrt{|h_{2L-2}|^2 + |g_{2L-2}|^2}$. In the case of $R' \neq 0$, c_{2L-3}, c_{2L-4}, a_1, and b_1 are given by

$$\begin{aligned} c_{2L-3} = \gamma' R', \quad c_{2L-4} = s c_{2L-3}, \\ a_1 = h_{2L-1} c_{2L-3}^{-1}, \quad b_1 = g_{2L-1} c_{2L-3}^{-1}, \end{aligned} \quad (12)$$

where γ' is a complex number with unit modulus and $s' = h_{2L-1}^{-1} h_{2L-2}$ if $h_{2L-1} \neq 0$, $s' = g_{2L-1}^{-1} g_{2L-2}$ otherwise. In the case of $R' = 0$ (in which case $R'' \neq 0$ holds from Eq. (6)), c_{2L-3}, c_{2L-4}, a_1, and b_1 are given by

$$\begin{aligned} c_{2L-3} = 0, \quad c_{2L-4} = \gamma'' R'', \\ a_1 = h_{2L-2} c_{2L-4}^{-1}, \quad b_1 = g_{2L-2} c_{2L-4}^{-1}, \end{aligned} \quad (13)$$

where γ'' is a complex number with unit modulus. Note that γ, γ', and γ'' can be any complex number with unit modulus.

From Eqs. (11), (12), and (13), we obatin A, $D^{(0)}$, and $D^{(L-1)}$. From $A \in \mathcal{U}_2$ and Eq. (10), $D^{(k)}$ with $0 < k < L-1$ can be obtained from $D^{(k)} = A^{-1}H^{(k)}$, from which we obtain C.

To complete the proof, it suffices to verify $A \in \mathcal{U}_2$. (Note that if $A \in \mathcal{U}_2$, then $C = [(I_{\frac{M}{2}} \otimes A)Q_M]^{-1}H_M \in \mathcal{U}_M$ obviously holds.) Since $H_M \in \mathcal{U}_M$ holds, we have $H_M^\dagger H_M = I_M$. Substituting Eq. (3) into this equation and using $M \geq 4L-2$, we have $[H^{(0)}]^\dagger H^{(L-1)} = O_2$, which gives $h_0^* h_{2L-1} + g_0^* g_{2L-1} = 0$. From this equation, we easily verify $A^\dagger A = I_2$, i.e., $A \in \mathcal{U}_2$. ∎

Note that the transform matrix W given by Eq. (5) corresponds to the QOWT based on a pyramid algorithm (PYA). The QOWT based on a packet algorithm (PAA), whose transform matrix W_{PAA} is given by

$$W_{\mathrm{PAA}} := W'_{\Xi-1}W'_{\Xi-2}\cdots W'_0,$$
$$W'_\xi := I_{N/N_\xi} \otimes (\Pi_{N_\xi}H_{N_\xi}),$$

can be implemented more efficiently than based on a PYA, while PYAs are more commonly used than PAAs in classical signal processing. Our results also readily apply to an efficient implementation for a PAA.

A. Example

As an example, we show the decomposition of the transform matrix H_M of the so-called Coiflet-6 wavelet [8]. This matrix has $L = 3$ and the scaling and wavelet filters with

$$h_0 = -0.072733, \quad h_1 = 0.337898, \quad h_2 = 0.852572,$$
$$h_3 = 0.384865, \quad h_4 = -0.072733, \quad h_5 = -0.015656,$$

and $g_t = (-1)^t h_{5-t}$ $(t \in \mathcal{I}_6)$. $H^{(k)}$ of Eq. (4) is

$$H^{(0)} = \begin{bmatrix} -0.072733 & 0.337898 \\ -0.015656 & 0.072733 \end{bmatrix},$$
$$H^{(1)} = \begin{bmatrix} 0.852572 & 0.384865 \\ 0.384865 & -0.852572 \end{bmatrix},$$
$$H^{(2)} = \begin{bmatrix} -0.072733 & -0.015656 \\ 0.337898 & 0.072733 \end{bmatrix},$$
$$H^{(k)} = O_2, \quad k \in \{3, \ldots, M/2-1\}.$$

Theorem 1 says that H_M with $M \geq 10$ can be decomposed as in Eq. (7). From the proof of Lemma 2, we obtain

$$A_2 = \begin{bmatrix} -0.977609 & -0.210431 \\ -0.210431 & 0.977609 \end{bmatrix},$$
$$A_1 = \begin{bmatrix} 0.935414 & 0.353553 \\ 0.353553 & -0.935414 \end{bmatrix},$$
$$A_0 = \begin{bmatrix} 0.977609 & 0.210431 \\ 0.210431 & -0.977609 \end{bmatrix}.$$

Indeed, H_M can be decomposed by $H_M = (I_{\frac{M}{2}} \otimes A_2)Q_M C$ [see Eq. (8)], where C is expressed by Eq. (9) with

$$C^{(0)} = \begin{bmatrix} 0.196840 & -0.914469 \\ 0.074398 & -0.345637 \end{bmatrix},$$
$$C^{(1)} = \begin{bmatrix} 0.345637 & 0.074398 \\ -0.914469 & -0.196840 \end{bmatrix},$$
$$C^{(k)} = O_2, \quad k \in \{2, \ldots, M/2-1\}.$$

C is also decomposed by $C = (I_{\frac{M}{2}} \otimes A_2)Q_M(I_{\frac{M}{2}} \otimes A_1)$.

IV. Implementation of undecimated QWTs

A. Formulation

We now turn our attention to the UWT using the same scaling and wavelet filters as in an OWT (for details see, e.g., [2]). For each scale $\xi \in \mathcal{I}_\Xi$, the UWT calculates the scaling coefficients $\boldsymbol{s}^{(\xi+1)}$ and the wavelet coefficients $\boldsymbol{w}^{(\xi+1)}$ using the following equation:

$$s_k^{(\xi+1)} = \frac{1}{\sqrt{2}}\sum_{t=0}^{2L-1} h_t s_{k+2^\xi t}^{(\xi)}, \quad w_k^{(\xi+1)} = \frac{1}{\sqrt{2}}\sum_{t=0}^{2L-1} g_t s_{k+2^\xi t}^{(\xi)}, \quad k \in \mathcal{I}_N,$$

where $\boldsymbol{s}^{(0)} := \boldsymbol{x}$. In the UWT, $\boldsymbol{s}^{(\xi+1)}$ and $\boldsymbol{w}^{(\xi+1)}$ are undecimated and belong to $\mathbb{C}^N$, whereas in the OWT these coefficients are decimated by a factor of two [see Eq. (1)]. The output of the UWT is $[\boldsymbol{w}^{(1)\mathsf{T}}, \ldots, \boldsymbol{w}^{(\Xi)\mathsf{T}}, \boldsymbol{s}^{(\Xi)\mathsf{T}}]^\mathsf{T} \in \mathbb{C}^{N(\Xi+1)}$. The UWT has the advantage of being shift-invariant, which means that if $[\boldsymbol{w}^{(1)\mathsf{T}}, \ldots, \boldsymbol{w}^{(\Xi)\mathsf{T}}, \boldsymbol{s}^{(\Xi)\mathsf{T}}]^\mathsf{T}$ is obtained by the UWT of the input signal $\boldsymbol{x}$, then $[Q_N^\dagger \boldsymbol{w}^{(1)\mathsf{T}}, \ldots, Q_N^\dagger \boldsymbol{w}^{(\Xi)\mathsf{T}}, Q_N^\dagger \boldsymbol{s}^{(\Xi)\mathsf{T}}]^\mathsf{T}$ is obtained by the UWT of $Q_N^\dagger \boldsymbol{x}$, where

$$Q_N^\dagger \boldsymbol{x} = [x_{N-1}, x_0, \ldots, x_{N-2}]^\mathsf{T}.$$

Shift invariant property is well known to provide good performance in various signal processing tasks. The unitary transformation represented by the transform matrix of the UWT is called a quantum UWT (QUWT).

At each level ξ, the undecimated transform is expressed by

$$\boldsymbol{\eta}_q^{(\xi+1)} = \frac{1}{\sqrt{2}}H_{N_\xi}\boldsymbol{s}_q^{(\xi)},$$
$$\boldsymbol{\eta}_{q+2^\xi}^{(\xi+1)} = \frac{1}{\sqrt{2}}H_{N_\xi}Q_{N_\xi}\boldsymbol{s}_q^{(\xi)} \tag{14}$$

for each $q \in \mathcal{I}_{2^\xi}$, where

$$\boldsymbol{\eta}_q^{(\xi+1)} := [s_q^{(\xi+1)}, w_q^{(\xi+1)}, s_{q+2^{\xi+1}}^{(\xi+1)}, w_{q+2^{\xi+1}}^{(\xi+1)}, s_{q+2\cdot 2^{\xi+1}}^{(\xi+1)}, w_{q+2\cdot 2^{\xi+1}}^{(\xi+1)}, \ldots]^\mathsf{T},$$
$$\boldsymbol{s}_q^{(\xi)} := [s_q^{(\xi)}, s_{q+2^\xi}^{(\xi)}, s_{q+2\cdot 2^\xi}^{(\xi)}, \ldots]^\mathsf{T},$$

$\boldsymbol{\eta}_q^{(\xi+1)}, \boldsymbol{\eta}_{q+2^\xi}^{(\xi+1)}, \boldsymbol{s}_q^{(\xi)} \in \mathbb{C}^{N_\xi}$ holds. Equation (14) can be rewritten as

$$\begin{bmatrix} \boldsymbol{\eta}_q^{(\xi+1)} \\ \boldsymbol{\eta}_{q+2^\xi}^{(\xi+1)} \end{bmatrix} = \tilde{H}_{2N_\xi}\begin{bmatrix} \boldsymbol{s}_q^{(\xi)} \\ \boldsymbol{0}_{N_\xi} \end{bmatrix},$$
$$\tilde{H}_{2N_\xi} := \frac{1}{\sqrt{2}}\begin{bmatrix} H_{N_\xi} & X_1 \\ H_{N_\xi}Q_{N_\xi} & X_2 \end{bmatrix}, \tag{15}$$

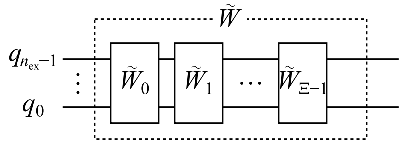

Fig. 3. Quantum circuit for $\tilde{W}$.

where X_1 and X_2 are any square matrices of order N_ξ such that $\tilde{H}_{2N_\xi} \in \mathcal{U}_{2N_\xi}$.

B. Implementation of QUWTs

Since the output of QUWT, $[\boldsymbol{w}^{(1)\mathsf{T}}, \ldots, \boldsymbol{w}^{(\Xi)\mathsf{T}}, \boldsymbol{s}^{(\Xi)\mathsf{T}}]^\mathsf{T} \in \mathbb{C}^{N(\Xi+1)}$, has $N(\Xi+1)$ elements, we can assume, without loss of generality, that the QUWT operates on $n_{\rm ex} := \lceil \log_2 N(\Xi+1) \rceil = n + \lceil \log_2(\Xi+1) \rceil$ qubits, where $\lceil x \rceil$ is the smallest integer that is not less than x. Let $N_{\rm ex} := 2^{n_{\rm ex}}$.

The QUWT is represented by the unitary matrix $\tilde{W} \in \mathcal{U}_{N_{\rm ex}}$, whose input signal is $\tilde{\boldsymbol{x}} := \boldsymbol{x} \oplus \boldsymbol{0}_{N_{\rm ex}-N} \in \mathbb{C}^{N_{\rm ex}}$. After some simple computation, we can see that $\tilde{W}$ is expressed by

$$\tilde{W} := \tilde{W}_{\Xi-1} \tilde{W}_{\Xi-2} \cdots \tilde{W}_0,$$

where

$$\begin{aligned}
\tilde{W}_\xi &:= \begin{cases} \Gamma^{(3)}_\xi \Gamma^{(2)} \Gamma^{(1)} \Phi_\xi, & \xi < \Xi - 1, \\ \Gamma^{(2)} \Gamma^{(1)} \Phi_\xi, & \xi = \Xi - 1, \end{cases} \\
\Gamma^{(3)}_\xi &:= \Pi^\dagger_{2^{\xi+2}} \otimes I_{N/2^{\xi+1}} \oplus I_E, \\
\Gamma^{(2)} &:= R \otimes I_N, \\
\Gamma^{(1)} &:= \Pi_{2N} \oplus I_E, \\
\Phi_\xi &:= I_{2^\xi} \otimes \tilde{H}_{2N_\xi} \oplus I_E
\end{aligned} \tag{16}$$

and $E := N_{\rm ex} - 2N$. $R \in \mathbb{C}^{N_R \times N_R}$ $(N_R := N_{\rm ex}/N)$ denotes the permutation matrix whose $(j+1, k+1)$-th element is

$$R_{j,k} := \begin{cases} \delta_{k,0}, & j = 0, \\ \delta_{k,j+1}, & 0 < j < N_R - 1, \\ \delta_{k,1}, & j = N_R - 1. \end{cases}$$

The scaling and wavelet coefficients are obtained by the matrix $\Phi_\xi \in \mathbb{C}^{N_{\rm ex} \times N_{\rm ex}}$. $\Gamma^{(3)}_\xi, \Gamma^{(2)}, \Gamma^{(1)} \in \mathbb{C}^{N_{\rm ex} \times N_{\rm ex}}$ are permutation matrices, which are used to rearrange the coefficients. Similar to W, $\tilde{W}$ is also completely characterized by the set of matrices $\{H^{(k)}\}_{k=0}^{L-1}$. The QUWT with the transform matrix $\tilde{W}$ can be implemented in the quantum circuit shown in Fig. 3, where the circuit for $\tilde{W}_\xi$ is shown in Fig. 4. As an example, Table I shows $\tilde{\boldsymbol{x}}$, $\tilde{W}_0 \tilde{\boldsymbol{x}}$, $\tilde{W}_1 \tilde{W}_0 \tilde{\boldsymbol{x}}$, and $\tilde{W} \tilde{\boldsymbol{x}} = \tilde{W}_2 \tilde{W}_1 \tilde{W}_0 \tilde{\boldsymbol{x}}$ for $N = 8$ and $\Xi = 3$, in which case $N_{\rm ex} = 32$ holds. From Eq. (16), if there exists an efficient implementation for Φ_ξ (i.e., $\tilde{H}_{2N_\xi}$), then the QUWT can be efficiently implemented.

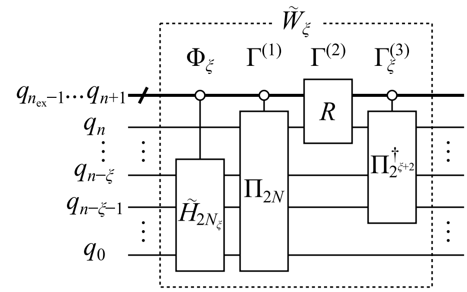

Fig. 4. Quantum circuit for $\tilde{W}_\xi$ (with $\xi < \Xi - 1$).

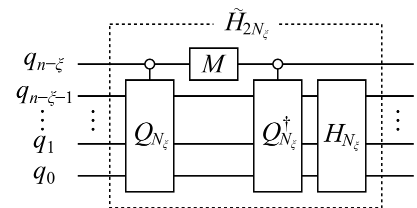

Fig. 5. Quantum circuit for $\tilde{H}_{2N_\xi}$.

We show that Φ_ξ can be implemented with a complexity of $O(n_{\rm ex})$. Let us substitute $X_1 := H_{N_\xi} Q^\dagger_{N_\xi}$ and $X_2 := -H_{N_\xi}$ into Eq. (15); we obtain

$$\tilde{H}_{2N_\xi} := \frac{1}{\sqrt{2}} \begin{bmatrix} H_{N_\xi} & H_{N_\xi} Q^\dagger_{N_\xi} \\ H_{N_\xi} Q_{N_\xi} & -H_{N_\xi} \end{bmatrix}.$$

We can easily verify $\tilde{H}_{2N_\xi} \in \mathcal{U}_{2N_\xi}$. $\tilde{H}_{2N_\xi}$ can be expressed by

$$\tilde{H}_{2N_\xi} = (I_2 \otimes H_{N_\xi})(Q^\dagger_{N_\xi} \oplus I_{N_\xi})(M \otimes I_{N_\xi})(Q_{N_\xi} \oplus I_{N_\xi}),$$

where $M := \frac{1}{\sqrt{2}} \left[\begin{smallmatrix} 1 & 1 \\ 1 & -1 \end{smallmatrix}\right]$. Thus, $\tilde{H}_{2N_\xi}$ can be efficiently implemented as shown in Fig. 5, which implies that Φ_ξ can be implemented with a complexity of $O(n_{\rm ex})$.

V. Conclusion

We proposed an efficient implementation of any QOWT and its undecimated version. The main result of this paper is that the block circulant matrix H_M can be decomposed into $L-1$ permutation matrices and L single-qubit unitary matrices, which allows us to implement the QOWT with a complexity of $O(n)$. We also showed an implementation of the QUWT with a complexity of $O(n_{\rm ex})$.

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TABLE I
THE QUWT FOR $N = 8$ AND $\Xi = 3$.

vectors	rows															
	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16
$\tilde{\boldsymbol{x}}$	x_0	x_1	x_2	x_3	x_4	x_5	x_6	x_7	0	0	0	0	0	0	0	0
$\tilde{W}_0\tilde{\boldsymbol{x}}$	$s_0^{(1)}$	$s_2^{(1)}$	$s_4^{(1)}$	$s_6^{(1)}$	0	0	0	0	$s_1^{(1)}$	$s_3^{(1)}$	$s_5^{(1)}$	$s_7^{(1)}$	0	0	0	0
$\tilde{W}_1\tilde{W}_0\tilde{\boldsymbol{x}}$	$s_0^{(2)}$	$s_4^{(2)}$	0	0	$s_2^{(2)}$	$s_6^{(2)}$	0	0	$s_1^{(2)}$	$s_5^{(2)}$	0	0	$s_3^{(2)}$	$s_7^{(2)}$	0	0
$\tilde{W}\tilde{\boldsymbol{x}}$	$s_0^{(3)}$	$s_4^{(3)}$	$s_2^{(3)}$	$s_6^{(3)}$	$s_1^{(3)}$	$s_5^{(3)}$	$s_3^{(3)}$	$s_7^{(3)}$	$w_0^{(1)}$	$w_2^{(1)}$	$w_4^{(1)}$	$w_6^{(1)}$	$w_1^{(1)}$	$w_3^{(1)}$	$w_5^{(1)}$	$w_7^{(1)}$

vectors	rows															
	17	18	19	20	21	22	23	24	25	26	27	28	29	30	31	32
$\tilde{\boldsymbol{x}}$	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
$\tilde{W}_0\tilde{\boldsymbol{x}}$	0	0	0	0	0	0	0	0	$w_0^{(1)}$	$w_2^{(1)}$	$w_4^{(1)}$	$w_6^{(1)}$	$w_1^{(1)}$	$w_3^{(1)}$	$w_5^{(1)}$	$w_7^{(1)}$
$\tilde{W}_1\tilde{W}_0\tilde{\boldsymbol{x}}$	$w_0^{(1)}$	$w_2^{(1)}$	$w_4^{(1)}$	$w_6^{(1)}$	$w_1^{(1)}$	$w_3^{(1)}$	$w_5^{(1)}$	$w_7^{(1)}$	$w_0^{(2)}$	$w_4^{(2)}$	$w_2^{(2)}$	$w_6^{(2)}$	$w_1^{(2)}$	$w_5^{(2)}$	$w_3^{(2)}$	$w_7^{(2)}$
$\tilde{W}\tilde{\boldsymbol{x}}$	$w_0^{(2)}$	$w_4^{(2)}$	$w_2^{(2)}$	$w_6^{(2)}$	$w_1^{(2)}$	$w_5^{(2)}$	$w_3^{(2)}$	$w_7^{(2)}$	$w_0^{(3)}$	$w_4^{(3)}$	$w_2^{(3)}$	$w_6^{(3)}$	$w_1^{(3)}$	$w_5^{(3)}$	$w_3^{(3)}$	$w_7^{(3)}$

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