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# A Note on Belavkin-Weighted Square-Root Measurement for Binary Pure State Ensembles

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***Abstract*—This article concerns the Belavkin-weighted square-root measurement (BWSRM) for binary pure state ensembles. For given weights of the BWSRM, a closed-form expression of the state distribution that makes the BWSRM Bayes-optimal is derived. Conversely, a closed-form expression of the optimal weights of the BWSRM is derived for given probability distribution of the states.**

## I. Introduction

It is well known that quantum detection theory plays an important role in quantum information science [1], [2]. This fundamental theory involves various types of decision strategies such as the Bayes strategy (which seeks the optimal measurement to achive the minimal average probability of error — or more generally, the minimum average cost for decision — when prior probabilities of the states are given), the minimax strategy (which is used when the prior probabilities are unknown), and so on. The development of quantum detection theory was initiated by the pioneering works for establishing the Bayes strategy in the framework of quantum measurement, which were led by Helstrom [3], Yuen [5], [6], and Holevo [4]. In parallel to their works, Belavkin introduced a constractive method for finding quantum measurements [7], [8]. Today, Belavkin's works related to this method is recognized as early studies of the square-root measurement (SRM).

The family of the SRM [7], [8], [9], [10] provides useful tools for analyzing various problems appeard in quantum information theory (*e.g.* [11], [12], [13], [14], [15], [16], [17]). In particular, the SRM is one of key parts in the proof of the quantum channel coding theorems [10], [18], [19]. The family of the SRMs is classfied into several subgroups. In this article, we call a measurement scheme given by the original description of Belavkin's the Belavkin-weighted square-root measurement (BWSRM) (See [20]), and a simplified non-weighted one the SRM simply.

The basic properties of the family of the SRMs have been investigated by many authors so far (*e.g.* [20], [21], [22], [23], [24], [25]), in which the main topics are its asymptotic behavior and its relation to the Bayes strategy. As for the relationship between the BWSRM and the Bayes-optimal measurement, which is defined to be the optimal measusmrent for achieving the minimal average probability of error when the prior probabilities of the states are given in this article, some remarkable results related to the optimal weights of the BWSRM for given probability distribution of the states [23] and the optimal probabiliy distribution for given weights [20], [25] have been reported. However, almost results are given in a general description, not in a concrete case. By this reason, we aim to give a small concrete example in this article, so that the case of binary pure state ensembles is considered.

## II. Square-root measurement and Belavkin weighted square-root measurement

Suppose $\{|\psi_m\rangle : m = 1, 2, \ldots, M\}$ is a collection of $M$ linearly independent pure states. For each state $|\psi_m\rangle$, we assign a prior probability $p_m > 0$. The collection of pairs $(|\psi_m\rangle, p_m)$ forms an $M$-ary pure state ensemble $\mathcal{E}$.

The SRM $\mathit{\Pi}^\circ$ for $\mathcal{E}$ is defined as follows [9]:

$$\mathit{\Pi}^\circ = \{\hat{\mathit{\Pi}}^\circ_m = |\mu^\circ_m\rangle\langle\mu^\circ_m| : m = 1, 2, \ldots, M\} \quad (1)$$

with

$$|\mu^\circ_m\rangle = \left(\sum_{\ell=1}^{M} |\psi_\ell\rangle\langle\psi_\ell|\right)^{-1/2} |\psi_m\rangle. \quad (2)$$

Let $w_m > 0$ for $m = 1, 2, \ldots, M$. The BWSRM $\mathit{\Pi}^\bullet$ for $\mathcal{E}$ is defined as follows [7], [8]:

$$\mathit{\Pi}^\bullet = \{\hat{\mathit{\Pi}}^\bullet_m = |\mu^\bullet_m\rangle\langle\mu^\bullet_m| : m = 1, 2, \ldots, M\} \quad (3)$$

with

$$|\mu^\bullet_m\rangle = \left(\sum_{\ell=1}^{M} w_\ell |\psi_\ell\rangle\langle\psi_\ell|\right)^{-1/2} \sqrt{w_m}\, |\psi_m\rangle. \quad (4)$$

## III. BWSRM for binary pure state ensembles

Suppose two non-orthogonal states $|\psi_1\rangle$ and $|\psi_2\rangle$ are given, in which the inner product of the two states is $\langle\psi_1|\psi_2\rangle = e^{\jmath\theta}\kappa$ with $0 < \kappa < 1$, $0 \le \theta < 2\pi$, and $\jmath = \sqrt{-1}$. The measurement vectors $|\mu^\circ_1\rangle$ and $|\mu^\circ_2\rangle$ of the SRM $\mathit{\Pi}^\circ$ for a binary pure state ensemble are respectively given by

$$|\mu^\circ_1\rangle = A^\circ|\psi_1\rangle + B^{\circ *}|\psi_2\rangle, \quad (5)$$
$$|\mu^\circ_2\rangle = B^\circ|\psi_1\rangle + A^\circ|\psi_2\rangle, \quad (6)$$

with

$$A^\circ = \frac{1}{2}\left(\frac{1}{\sqrt{1-\kappa}}+\frac{1}{\sqrt{1+\kappa}}\right) = \sqrt{\frac{1+\sqrt{1-\kappa^2}}{2(1-\kappa^2)}}, \quad (7)$$

$$B^\circ = -\frac{1}{2}\mathrm{e}^{\jmath\theta}\left(\frac{1}{\sqrt{1-\kappa}}-\frac{1}{\sqrt{1+\kappa}}\right) = -\mathrm{e}^{\jmath\theta}\sqrt{\frac{1-\sqrt{1-\kappa^2}}{2(1-\kappa^2)}}, \quad (8)$$

where $*$ stands for the complrex conjugate. As expected from a general nature of the SRM, the set $\beta^\circ = \{|\mu_1^\circ\rangle, |\mu_2^\circ\rangle\}$ is a orthonormal basis of the space $\mathcal{H}_2 = \mathrm{span}(\{|\psi_1\rangle, |\psi_2\rangle\})$, because $\kappa \neq 1$. Based on this fact, the states can be represented in the basis $\beta^\circ$ as follows:

$$|\psi_1\rangle = C^\circ|\mu_1^\circ\rangle + D^{\circ *}|\mu_2^\circ\rangle, \quad (9)$$

$$|\psi_2\rangle = D^\circ|\mu_1^\circ\rangle + C^\circ|\mu_2^\circ\rangle, \quad (10)$$

with

$$C^\circ = \frac{1}{2}\left(\sqrt{1+\kappa}+\sqrt{1-\kappa}\right), \quad (11)$$

$$D^\circ = -\frac{1}{2}\mathrm{e}^{\jmath\theta}\left(\sqrt{1+\kappa}-\sqrt{1-\kappa}\right). \quad (12)$$

From these expressions, the corresponding density operators $\hat{\rho}_1 = |\psi_1\rangle\langle\psi_1|$ and $\hat{\rho}_2 = |\psi_2\rangle\langle\psi_2|$ can be written as

$$\begin{aligned}\hat{\rho}_1 &= C^{\circ 2}|\mu_1^\circ\rangle\langle\mu_1^\circ| + C^\circ D^\circ|\mu_1^\circ\rangle\langle\mu_2^\circ| + C^\circ D^{\circ *}|\mu_2^\circ\rangle\langle\mu_1^\circ| + |D^\circ|^2|\mu_2^\circ\rangle\langle\mu_2^\circ| \\ &= \frac{1+\sqrt{1-\kappa^2}}{2}|\mu_1^\circ\rangle\langle\mu_1^\circ| + \frac{\kappa\mathrm{e}^{\jmath\theta}}{2}|\mu_1^\circ\rangle\langle\mu_2^\circ| + \frac{\kappa\mathrm{e}^{-\jmath\theta}}{2}|\mu_2^\circ\rangle\langle\mu_1^\circ| + \frac{1-\sqrt{1-\kappa^2}}{2}|\mu_2^\circ\rangle\langle\mu_2^\circ|,\end{aligned} \quad (13)$$

and

$$\begin{aligned}\hat{\rho}_2 &= |D^\circ|^2|\mu_1^\circ\rangle\langle\mu_1^\circ| + C^\circ D^\circ|\mu_1^\circ\rangle\langle\mu_2^\circ| + C^\circ D^{\circ *}|\mu_2^\circ\rangle\langle\mu_1^\circ| + C^{\circ 2}|\mu_2^\circ\rangle\langle\mu_2^\circ| \\ &= \frac{1-\sqrt{1-\kappa^2}}{2}|\mu_1^\circ\rangle\langle\mu_1^\circ| + \frac{\kappa\mathrm{e}^{\jmath\theta}}{2}|\mu_1^\circ\rangle\langle\mu_2^\circ| + \frac{\kappa\mathrm{e}^{-\jmath\theta}}{2}|\mu_2^\circ\rangle\langle\mu_1^\circ| + \frac{1+\sqrt{1-\kappa^2}}{2}|\mu_2^\circ\rangle\langle\mu_2^\circ|.\end{aligned} \quad (14)$$

Hence the sum of the density operators — the Gram operator — is

$$\begin{aligned}\hat{G} &= |\psi_1\rangle\langle\psi_1| + |\psi_2\rangle\langle\psi_2| \\ &= |\mu_1^\circ\rangle\langle\mu_1^\circ| + \kappa\mathrm{e}^{\jmath\theta}|\mu_1^\circ\rangle\langle\mu_2^\circ| + \kappa\mathrm{e}^{-\jmath\theta}|\mu_2^\circ\rangle\langle\mu_1^\circ| + |\mu_2^\circ\rangle\langle\mu_2^\circ| \\ &= \langle\psi_1|\psi_1\rangle|\mu_1^\circ\rangle\langle\mu_1^\circ| + \langle\psi_1|\psi_2\rangle|\mu_1^\circ\rangle\langle\mu_2^\circ| + \langle\psi_2|\psi_1\rangle|\mu_2^\circ\rangle\langle\mu_1^\circ| + \langle\psi_2|\psi_2\rangle|\mu_2^\circ\rangle\langle\mu_2^\circ|,\end{aligned} \quad (15)$$

which explains that the matrix representation of $\hat{G}$ in the basis $\beta^\circ$ is

$$[\hat{G}]_{\beta^\circ} = \begin{bmatrix} 1 & \mathrm{e}^{\jmath\theta}\kappa \\ \mathrm{e}^{-\jmath\theta}\kappa & 1 \end{bmatrix}. \quad (16)$$

On the other hand, the measurement vectors $|\mu_1^\bullet\rangle$ and $|\mu_2^\bullet\rangle$ of the BWSRM $\Pi^\bullet$ for a binary pure state ensemble are respectively given by

$$|\mu_1^\bullet\rangle = A_1^\bullet|\psi_1\rangle + B_1^\bullet|\psi_2\rangle, \quad (17)$$

$$|\mu_2^\bullet\rangle = B_2^\bullet|\psi_1\rangle + A_2^\bullet|\psi_2\rangle, \quad (18)$$

with

$$A_1^\bullet = \frac{\sqrt{(1-\kappa^2)w_1}+\sqrt{w_2}}{\sqrt{1-\kappa^2}\sqrt{w_1+w_2+2\sqrt{(1-\kappa^2)w_1w_2}}} = \frac{\sqrt{(1-\kappa^2)q_1}+\sqrt{q_2}}{\sqrt{1-\kappa^2}\sqrt{1+2\sqrt{(1-\kappa^2)q_1q_2}}}, \quad (19)$$

$$B_1^\bullet = \frac{-\mathrm{e}^{-\jmath\theta}\kappa\sqrt{w_2}}{\sqrt{1-\kappa^2}\sqrt{w_1+w_2+2\sqrt{(1-\kappa^2)w_1w_2}}} = \frac{-\mathrm{e}^{-\jmath\theta}\kappa\sqrt{q_2}}{\sqrt{1-\kappa^2}\sqrt{1+2\sqrt{(1-\kappa^2)q_1q_2}}}, \quad (20)$$

$$B_2^\bullet = \frac{-\mathrm{e}^{\jmath\theta}\kappa\sqrt{w_1}}{\sqrt{1-\kappa^2}\sqrt{w_1+w_2+2\sqrt{(1-\kappa^2)w_1w_2}}} = \frac{-\mathrm{e}^{\jmath\theta}\kappa\sqrt{q_1}}{\sqrt{1-\kappa^2}\sqrt{1+2\sqrt{(1-\kappa^2)q_1q_2}}}, \quad (21)$$

$$A_1^\bullet = \frac{\sqrt{w_1}+\sqrt{(1-\kappa^2)w_2}}{\sqrt{1-\kappa^2}\sqrt{w_1+w_2+2\sqrt{(1-\kappa^2)w_1w_2}}} = \frac{\sqrt{q_1}+\sqrt{(1-\kappa^2)q_2}}{\sqrt{1-\kappa^2}\sqrt{1+2\sqrt{(1-\kappa^2)q_1q_2}}}, \quad (22)$$

where $q_1$ and $q_2$ are defined by

$$q_1 = \frac{w_1}{w_1+w_2}, \qquad q_2 = \frac{w_2}{w_1+w_2}. \quad (23)$$

The vectors $|\mu_1^\bullet\rangle$ and $|\mu_2^\bullet\rangle$ form another orthonormal basis for $\mathcal{H}_2$, $\beta^\bullet = \{|\mu_1^\bullet\rangle, |\mu_2^\bullet\rangle\}$. Therefore, we obtain another representation of the states as follows:

$$|\psi_1\rangle = C_1^\bullet|\mu_1^\bullet\rangle + D_1^\bullet|\mu_2^\bullet\rangle, \quad (24)$$

$$|\psi_2\rangle = D_2^\bullet|\mu_1^\bullet\rangle + C_2^\bullet|\mu_2^\bullet\rangle, \quad (25)$$

with

$$C_1^\bullet = \frac{\sqrt{w_1}+\sqrt{(1-\kappa^2)w_2}}{\sqrt{w_1+w_2+2\sqrt{(1-\kappa^2)w_1w_2}}} = \frac{\sqrt{q_1}+\sqrt{(1-\kappa^2)q_2}}{\sqrt{1+2\sqrt{(1-\kappa^2)q_1q_2}}}, \quad (26)$$

$$D_1^\bullet = \frac{e^{-\jmath\theta}\kappa\sqrt{w_2}}{\sqrt{w_1+w_2+2\sqrt{(1-\kappa^2)w_1w_2}}} = \frac{e^{-\jmath\theta}\kappa\sqrt{q_2}}{\sqrt{1+2\sqrt{(1-\kappa^2)q_1q_2}}}, \tag{27}$$

$$D_2^\bullet = \frac{e^{\jmath\theta}\kappa\sqrt{w_1}}{\sqrt{w_1+w_2+2\sqrt{(1-\kappa^2)w_1w_2}}} = \frac{e^{-\jmath\theta}\kappa\sqrt{q_1}}{\sqrt{1+2\sqrt{(1-\kappa^2)q_1q_2}}}, \tag{28}$$

$$C_2^\bullet = \frac{\sqrt{(1-\kappa^2)w_1}+\sqrt{w_2}}{\sqrt{w_1+w_2+2\sqrt{(1-\kappa^2)w_1w_2}}} = \frac{\sqrt{(1-\kappa^2)q_1}+\sqrt{q_2}}{\sqrt{1+2\sqrt{(1-\kappa^2)q_1q_2}}}. \tag{29}$$

Then the density operators $\hat{\rho}_1$ and $\hat{\rho}_2$ can be written as

$$\begin{aligned}\hat{\rho}_1 &= C_1^{\bullet 2}|\mu_1^\bullet\rangle\langle\mu_1^\bullet| + C_1^\bullet D_1^{\bullet *}|\mu_1^\bullet\rangle\langle\mu_2^\bullet| + C_1^\bullet D_1^\bullet|\mu_2^\bullet\rangle\langle\mu_1^\bullet| + |D_1^\bullet|^2|\mu_2^\bullet\rangle\langle\mu_2^\bullet| \\ &= \frac{\left(w_1+(1-\kappa^2)w_2 + 2\sqrt{(1-\kappa^2)w_1w_2}\right)}{w_1+w_2+2\sqrt{(1-\kappa^2)w_1w_2}}|\mu_1^\bullet\rangle\langle\mu_1^\bullet| \\ &\quad + \frac{e^{\jmath\theta}\kappa\left(\sqrt{1-\kappa^2}\,w_2+\sqrt{w_1w_2}\right)}{w_1+w_2+2\sqrt{(1-\kappa^2)w_1w_2}}|\mu_1^\bullet\rangle\langle\mu_2^\bullet| \\ &\quad + \frac{e^{-\jmath\theta}\kappa\left(\sqrt{1-\kappa^2}\,w_2+\sqrt{w_1w_2}\right)}{w_1+w_2+2\sqrt{(1-\kappa^2)w_1w_2}}|\mu_2^\bullet\rangle\langle\mu_1^\bullet| \\ &\quad + \frac{\kappa^2 w_2}{w_1+w_2+2\sqrt{(1-\kappa^2)w_1w_2}}|\mu_2^\bullet\rangle\langle\mu_2^\bullet|,\end{aligned} \tag{30}$$

and

$$\begin{aligned}\hat{\rho}_2 &= |D_2^\bullet|^2|\mu_1^\bullet\rangle\langle\mu_1^\bullet| + C_2^\bullet D_2^\bullet|\mu_1^\bullet\rangle\langle\mu_2^\bullet| + C_2^\bullet D_2^{\bullet *}|\mu_2^\bullet\rangle\langle\mu_1^\bullet| + C_2^{\bullet 2}|\mu_2^\bullet\rangle\langle\mu_2^\bullet| \\ &= \frac{\kappa^2 w_1}{w_1+w_2+2\sqrt{(1-\kappa^2)w_1w_2}}|\mu_1^\bullet\rangle\langle\mu_1^\bullet| \\ &\quad + \frac{e^{\jmath\theta}\kappa\left(\sqrt{1-\kappa^2}\,w_1+\sqrt{w_1w_2}\right)}{w_1+w_2+2\sqrt{(1-\kappa^2)w_1w_2}}|\mu_1^\bullet\rangle\langle\mu_2^\bullet| \\ &\quad + \frac{e^{-\jmath\theta}\kappa\left(\sqrt{1-\kappa^2}\,w_1+\sqrt{w_1w_2}\right)}{w_1+w_2+2\sqrt{(1-\kappa^2)w_1w_2}}|\mu_2^\bullet\rangle\langle\mu_1^\bullet| \\ &\quad + \frac{\left((1-\kappa^2)w_1+w_2 + 2\sqrt{(1-\kappa^2)w_1w_2}\right)}{w_1+w_2+2\sqrt{(1-\kappa^2)w_1w_2}}|\mu_2^\bullet\rangle\langle\mu_2^\bullet|.\end{aligned} \tag{31}$$

Therefore, the weighted sum of the density operators with weights $w_1$ and $w_2$ is

$$\begin{aligned}w_1\hat{\rho}_1 + w_2\hat{\rho}_2 &= w_1\,|\mu_1^\bullet\rangle\langle\mu_1^\bullet| + e^{\jmath\theta}\kappa\sqrt{w_1w_2}\,|\mu_1^\bullet\rangle\langle\mu_2^\bullet| + e^{-\jmath\theta}\kappa\sqrt{w_1w_2}\,|\mu_2^\bullet\rangle\langle\mu_1^\bullet| + w_2\,|\mu_2^\bullet\rangle\langle\mu_2^\bullet| \\ &= \sqrt{w_1}\langle\psi_1|\psi_1\rangle\sqrt{w_1}\,|\mu_1^\bullet\rangle\langle\mu_1^\bullet| + \sqrt{w_1}\langle\psi_1|\psi_2\rangle\sqrt{w_2}\,|\mu_1^\bullet\rangle\langle\mu_2^\bullet| \\ &\quad + \sqrt{w_2}\langle\psi_2|\psi_1\rangle\sqrt{w_1}\,|\mu_2^\bullet\rangle\langle\mu_1^\bullet| + \sqrt{w_2}\langle\psi_2|\psi_2\rangle\sqrt{w_2}\,|\mu_2^\bullet\rangle\langle\mu_2^\bullet|.\end{aligned} \tag{32}$$

This means that the matrix representation of the weighted sum of the density operators in the basis $\beta^\bullet$ is

$$[w_1\hat{\rho}_1 + w_2\hat{\rho}_2]_{\beta^\bullet} = \begin{bmatrix} w_1 & e^{\jmath\theta}\kappa\sqrt{w_1w_2} \\ e^{-\jmath\theta}\kappa\sqrt{w_1w_2} & w_2 \end{bmatrix}. \tag{33}$$

Comparing the system of Eqs. (9) and (10) with that of Eqs. (24) and (25), we have the following rules for the change of cordinates.

$$|\mu_1^\bullet\rangle = E|\mu_1^\circ\rangle + F^*|\mu_2^\circ\rangle, \tag{34}$$
$$|\mu_2^\bullet\rangle = -F|\mu_1^\circ\rangle + E|\mu_2^\circ\rangle, \tag{35}$$

and

$$|\mu_1^\circ\rangle = E|\mu_1^\bullet\rangle - F^*|\mu_2^\bullet\rangle, \tag{36}$$
$$|\mu_2^\circ\rangle = F|\mu_1^\bullet\rangle + E|\mu_2^\bullet\rangle, \tag{37}$$

where

$$\begin{aligned}E &= \frac{\sqrt{1+\kappa}+\sqrt{1-\kappa}}{2} \times \frac{\sqrt{w_1}+\sqrt{w_2}}{\sqrt{w_1+w_2+2\sqrt{(1-\kappa^2)w_1w_2}}} \\ &= \frac{\sqrt{1+\kappa}+\sqrt{1-\kappa}}{2} \times \frac{\sqrt{q_1}+\sqrt{q_2}}{\sqrt{1+2\sqrt{(1-\kappa^2)q_1q_2}}},\end{aligned} \tag{38}$$

$$\begin{aligned}F &= e^{\jmath\theta}\frac{\sqrt{1+\kappa}-\sqrt{1-\kappa}}{2} \times \frac{\sqrt{w_1}-\sqrt{w_2}}{\sqrt{w_1+w_2+2\sqrt{(1-\kappa^2)w_1w_2}}} \\ &= e^{\jmath\theta}\frac{\sqrt{1+\kappa}-\sqrt{1-\kappa}}{2} \times \frac{\sqrt{q_1}-\sqrt{q_2}}{\sqrt{1+2\sqrt{(1-\kappa^2)q_1q_2}}}.\end{aligned} \tag{39}$$

From the rules mentioned above, we see that the SRM $\Pi^\circ$ and the BWSRM $\Pi^\bullet$ for the same binary pure state ensemble are identical if and only if $w_1 = w_2$.

The average probability of error by the BWSRM $\Pi^\bullet$ for a binary pure state ensemble is

$$\bar{P}_{\rm e}^\bullet(p_1,p_2) = \frac{(p_1w_2+p_2w_1)\kappa^2}{w_1+w_2+2\sqrt{w_1w_2(1-\kappa^2)}} = \frac{(p_1q_2+p_2q_1)\kappa^2}{1+2\sqrt{q_1q_2(1-\kappa^2)}}. \quad (40)$$

(*cf.* Eq.(14) of [20]).

## IV. Bayes-optimal measurement and BWSRM in the case of binary pure state discrimination

Given $(p_1,p_2)$, the measurement vectors $|\mu_1^{\rm B}\rangle$ and $|\mu_2^{\rm B}\rangle$ of the Bayes-optimal POVM $\Pi^{\rm B} = \{\hat{\Pi}_1^{\rm B} = |\mu_1^{\rm B}\rangle\langle\mu_1^{\rm B}|, \hat{\Pi}_2^{\rm B} = |\mu_2^{\rm B}\rangle\langle\mu_2^{\rm B}|\}$ can be directly obtained by solving the eigenvalue problem of a subtruction form of the two density matrix representations of the correspongong pure states in an arbitrarily chosen basis accoding to the literatures [1], [3]. Here we take the following expressions of the vectors $|\mu_1^{\rm B}\rangle$ and $|\mu_2^{\rm B}\rangle$ by means of the SRM vectors $|\mu_1^\circ\rangle$ and $|\mu_2^\circ\rangle$ :

$$|\mu_1^{\rm B}\rangle = u_{11}|\mu_1^\circ\rangle + u_{12}|\mu_2^\circ\rangle, \quad (41)$$
$$|\mu_2^{\rm B}\rangle = u_{21}|\mu_1^\circ\rangle + u_{22}|\mu_2^\circ\rangle, \quad (42)$$

where

$$u_{11} = \frac{\Lambda+(1+\zeta)\sqrt{1-\kappa^2}}{\sqrt{2\Lambda^2+2(1+\zeta)\Lambda\sqrt{1-\kappa^2}}} = u_{22}, \quad (43)$$
$$u_{21} = \frac{{\rm e}^{\jmath\theta}\kappa(1-\zeta)}{\sqrt{2\Lambda^2+2(1+\zeta)\Lambda\sqrt{1-\kappa^2}}} = -u_{12}^*, \quad (44)$$

and

$$\Lambda = \sqrt{(1+\zeta)^2-4\zeta\kappa^2}, \qquad \zeta = p_1/p_2. \quad (45)$$

Substituting Eqs. (5) and (6) into Eqs. (41) and (42),

$$|\mu_1^{\rm B}\rangle = (u_{11}A^\circ+u_{12}B^\circ)|\psi_1\rangle + (u_{11}B^{\circ *}+u_{12}A^\circ)|\psi_2\rangle, \quad (46)$$
$$|\mu_2^{\rm B}\rangle = (u_{21}A^\circ+u_{22}B^\circ)|\psi_1\rangle + (u_{21}B^{\circ *}+u_{22}A^\circ)|\psi_2\rangle. \quad (47)$$

Further, Eqs. (36) and (37) yield

$$|\mu_1^{\rm B}\rangle = (u_{11}E+u_{12}F)|\mu_1^\bullet\rangle + (-u_{11}F^*+u_{12}E)|\mu_2^\bullet\rangle, \quad (48)$$
$$|\mu_2^{\rm B}\rangle = (u_{21}E+u_{22}F)|\mu_1^\bullet\rangle + (-u_{21}F^*+u_{22}E)|\mu_2^\bullet\rangle, \quad (49)$$

which are the expressions of the vectors $|\mu_1^{\rm B}\rangle$ and $|\mu_2^{\rm B}\rangle$ by means of the BWSRM vectors $|\mu_1^\bullet\rangle$ and $|\mu_2^\bullet\rangle$.

The POVM $\Pi^{\rm B} = \{|\mu_1^{\rm B}\rangle\langle\mu_1^{\rm B}|, |\mu_2^{\rm B}\rangle\langle\mu_2^{\rm B}|\}$ of course provides the well-known formula of the minimum average probability of error for the discrimination of two pure states in accordance with the Bayes strategy [1], [3]:

$$\bar{P}_{\rm e}^{\rm B}(p_1,p_2) = \frac{1}{2}\left(1-\sqrt{1-4p_1p_2\kappa^2}\right). \quad (50)$$

This is independent from the choise of the basis to express the vectors $|\mu_1^{\rm B}\rangle$ and $|\mu_2^{\rm B}\rangle$. Letting

$$p_1 = \frac{1}{2}+\frac{\epsilon}{2}, \quad p_2 = \frac{1}{2}-\frac{\epsilon}{2}, \quad (51)$$

with $-1<\epsilon<1$, Eq. (50) is arranged to the form

$$\bar{P}_{\rm e}^{\rm B}(\epsilon) = \frac{1}{2}\left(1-\sqrt{1-(1-\epsilon^2)\kappa^2}\right). \quad (52)$$

### A. Case for given $(w_1,w_2)$

According to Mochon [23], the probability distribution that makes the BWSRM Bayes-optimal is formaly given by

$$p'_m = \frac{\mathcal{C}}{\langle\psi_m|\left(\sum_{\ell=1}^M w_\ell|\psi_\ell\rangle\langle\psi_\ell|\right)^{-1/2}|\psi_m\rangle}, \quad (53)$$

where $\mathcal{C}$ is a constant for normalization.

For a binary pure state ensemble, the optimal probabilites are given by

$$p'_1 = \frac{1}{2}+\frac{\Delta}{2}, \quad p'_2 = \frac{1}{2}-\frac{\Delta}{2} \quad (54)$$

with

$$\Delta = \frac{\sqrt{1-\kappa^2}(w_1-w_2)}{\sqrt{1-\kappa^2}(w_1+w_2)+2\sqrt{w_1w_2}} = \frac{\sqrt{1-\kappa^2}(q_1-q_2)}{\sqrt{1-\kappa^2}+2\sqrt{q_1q_2}}. \quad (55)$$

In fact, this distribution $(p'_1,p'_2)$ for given weights $w_1$ and $w_2$ analytically leads to

$$u_{11}E+u_{12}F = -u_{21}F^*+u_{22}E = 1, \quad (56)$$
$$-u_{11}F^*+u_{12}E = u_{21}E+u_{22}F = 0, \quad (57)$$

(See Appendix *A*) and hence

$$|\mu_1^{\rm B}\rangle = |\mu_1^\bullet\rangle, \qquad |\mu_2^{\rm B}\rangle = |\mu_2^\bullet\rangle, \quad (58)$$

for $(p'_1,p'_2)$. Therefore, we have

$$\bar{P}_{\rm e}^\bullet(p'_1,p'_2) = \bar{P}_{\rm e}^{\rm B}(p'_1,p'_2), \quad (59)$$

as expected.

Here let us verify Eqs. (54) and (55) numerically. Fig. 1 shows the error probability by Eq. (40) with Eqs. (54) and (55) at $\kappa = 0.3$ and 0.7, together with the corresponding minimum average probaility of error of Eq. (50). Fifty random samples generated from Eqs. (40), (54), and (55) are plotted for each $\kappa$, in which $w_1$ and $w_2$ are independetly chosen at random in the range $(0,100]$. A solid line is drawn by Eq. (50). All sample point is just on the mimimum average probability of error, which illustrates the optimality of $p'_1$ and $p'_2$ when $w_1$ and $w_2$ are given.

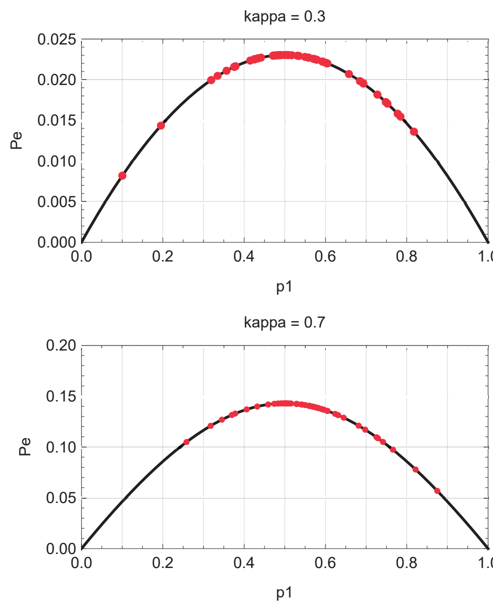


Fig. 1. Case for given $(w_1, w_2)$. $\kappa = 0.3$:top. $\kappa = 0.7$:bottom.

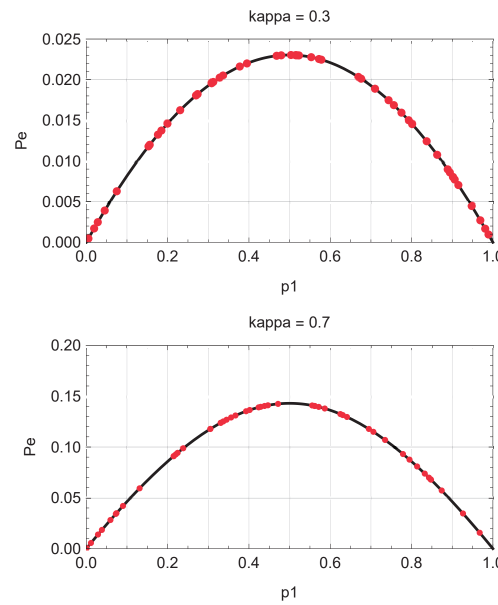


Fig. 2. Case for given $(p_1, p_2)$. $\kappa = 0.3$:top. $\kappa = 0.7$:bottom.

### *B. Case for given* $(p_1, p_2)$

According to the literatures [20] and [25], the optimal weights for given dstribution of the states are given by

$$w'_m = \langle \psi_m | \hat{\Pi}^{\mathrm{B}}_m | \psi_m \rangle p_m^2. \tag{60}$$

For a binary pure state ensemble, we have

$$w'_1 = \left( \frac{1}{2} + \frac{(1+\zeta) - 2\kappa^2}{2\Lambda} \right) p_1^2, \tag{61}$$

$$w'_2 = \left( \frac{1}{2} + \frac{(1+\zeta) - 2\zeta\kappa^2}{2\Lambda} \right) p_2^2. \tag{62}$$

Like in the previous case, these weights $w'_1$ and $w'_2$ yield

$$|\mu_1^{\mathrm{B}}\rangle = |\mu_1^{\bullet}\rangle|_{w'_1, w'_2}\,, \qquad |\mu_2^{\mathrm{B}}\rangle = |\mu_2^{\bullet}\rangle|_{w'_1, w'_2}\,, \tag{63}$$

and

$$\bar{P}_{\mathrm{e}}^{\bullet}(p_1, p_2)|_{w'_1, w'_2} = \bar{P}_{\mathrm{e}}^{\mathrm{B}}(p_1, p_2), \tag{64}$$

when $(p_1, p_2)$ is given.

Fig. 2 shows a numerical calculation result of the error probability by Eq. (40) with Eqs. (61) and (62) at $\kappa = 0.3$ and $0.7$, together with that of the minimum average probability of error of Eq. (50). Fifty random samples are plotted for each $\kappa$. This illustrates the optimality of $w'_1$ and $w'_2$ when $p_1$ and $p_2$ are given.

## V. Conclusion

We investigated some relationship among the square-root measurement (SRM), the Belavkin-weighted square-root measurement (BWSRM), and Bayes-optimal measurement in the case of binary pure state ensembles. Based on Mochon's result, a closed-form expression of the state distribution that makes the BWSRM Bayes-optimal was derived when the weights of the BWSRM are given. Conversely, a closed-form expression of the optimal weights of the BWSRM was derived when the probability distribution of the states is given, with the help of the preceding works by Tyson and by Łuczak and Wieczorek. These results give simple concrete examples of relation between the BWSRM and the Bayes-optimal measurement. More general analysis for seeking closed-form expressions such that for multiple state cases having more than three states ramains for future work.

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## APPENDIX

### A. Outline of the verification of Eqs. (56) and (57)

Both (57) and (56) can be obtained from some lengthy but straightforward algebra. Here we give a brief ouline of the verification of Eqs. (56) and (57).

First let us consider Eq. (56). Substituting Eq. (55) into $u_{11}E+u_{12}F$ and $-u_{21}F^*+u_{22}E$, which are coefficients appeared in Eqs. (51) and (52), we have

$$u_{11}E+u_{12}F=-u_{21}F^*+u_{22}E=\frac{\mathcal{N}_1}{\mathcal{D}_1},$$

where

$$\begin{aligned}\mathcal{N}_1 &= (\sqrt{1+\kappa}+\sqrt{1-\kappa})\\ &\quad\times(1+\sqrt{1-(1-\Delta^2)\kappa^2})\\ &\quad\quad\times(\sqrt{w_1}+\sqrt{w_2})\\ &\quad-\kappa(\sqrt{1+\kappa}-\sqrt{1-\kappa})\\ &\quad\quad\times\big((1-\Delta)\sqrt{w_1}+(1+\Delta)\sqrt{w_2}\big),\\ \mathcal{D}_1 &= 2\sqrt{2}\left(1-(1-\Delta^2)\kappa^2\right)^{\frac{1}{4}}\\ &\quad\times\sqrt{\sqrt{1-\kappa^2}+\sqrt{1-(1-\Delta^2)\kappa^2}}\\ &\quad\quad\times\sqrt{w_1+w_2+2\sqrt{(1-\kappa^2)w_1w_2}}.\end{aligned}$$

After lengthly algebra, we obtain

$$\frac{\mathcal{N}_1}{\mathcal{D}_1}=\frac{\sqrt{1+\kappa}+\sqrt{1-\kappa}}{\sqrt{2+2\sqrt{1-\kappa^2}}}=1.$$

In the middle, we have used the following facts to simplify the form of $\mathcal{N}_1/\mathcal{D}_1$:

$$\begin{aligned}&\left(\sqrt{1-\kappa^2}(w_1+w_2)2\sqrt{w_1w_2}\right)^2\\ &\quad-4\kappa^2\sqrt{w_1w_2}\\ &\quad\quad\times\left(\sqrt{1-\kappa^2}(w_1+w_2)+(2-\kappa^2)\sqrt{w_1w_2}\right)\\ =&\left(\sqrt{1-\kappa^2}(w_1+w_2)+2(1-\kappa^2)\sqrt{w_1w_2}\right)^2\end{aligned}$$

and $\sqrt{1-\kappa^2}(w_1+w_2)+2(1-\kappa^2)\sqrt{w_1w_2}>0$.

Next we consider Eq. (57). Substituting Eq. (55) into $-u_{11}F^*+u_{12}E$ and $u_{21}E+w_{22}F$, respectively, we have

$$e^{\jmath\theta}(-u_{11}F^*+u_{12}E)=e^{-\jmath\theta}(u_{21}E+w_{22}F)=\frac{\mathcal{N}_2}{\mathcal{D}_2},$$

where

$$\begin{aligned}\mathcal{N}_2 &= \Delta\kappa\big(\sqrt{1+\kappa}+\sqrt{1-\kappa}\big)\big(\sqrt{w_1}+\sqrt{w_2}\big)\\ &\quad-\big(\sqrt{1+\kappa}-\sqrt{1-\kappa}\big)\\ &\quad\quad\times\big(\sqrt{1-\kappa^2}+\sqrt{1-(1-\Delta^2)\kappa^2}\big)\\ &\quad\quad\quad\times\big(\sqrt{w_1}-\sqrt{w_2}\big),\\ \mathcal{D}_2 &= 2\sqrt{2}\sqrt{\begin{array}{l}1-(1-\Delta^2)\kappa^2\\ \quad+\sqrt{(1-\kappa^2)(1-(1-\Delta^2)\kappa^2)}\end{array}}\\ &\quad\times\sqrt{w_1+w_2+2\sqrt{(1-\kappa^2)w_1w_2}}.\end{aligned}$$

Since $0<1-\Delta^2<1$, we observe $\mathcal{D}_2>0$. Therefore, our task is to verify whether $\mathcal{N}_2$ vanishes or not. But, the straitforward calculation yields $\mathcal{N}_2=0$.