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Abstract—We compute the expurgated bound for a continuous classical-quantum channel with squeezed states numerically.

I. Introduction

The reliability function $E(R)$ is known as an index of asymptotical channel performance. It is a function of the communication rate R representing the speed of exponential decay of the error probability at the rate. That is, the error probability can be approximated by $\exp(-nE(R))$ when code length n takes a sufficiently large value. Quantum coding theorems for the reliability function were established by Holevo. On the analogy from the classical case, Holevo defined the random coding bound $E_r(R)$ and the expurgated bound $E_{ex}(R)$ based on the quantum channel coding, and proved that these give the lower bounds for the reliability function $E(R)$ truly [1]. Then Holevo proved the expurgated bound also holds in the mixed state case [4]. Moreover he extended these results to a continuous channel with constrained inputs. In addition the expurgated bound for coherent states with classical white Gaussian noise was computed analytically [5].

In this paper we compute the expurgated bound for a classical-quantum channel with squeezed states numerically. The expurgated bound has good performance at low rates compared to the random coding bound. The effect of using squeezed state on communication has been evaluated in several papers. In [3], we computed the capacity of the communication using squeezed states and showed that it cannot exceed the capacity in the case of using coherent states. On the other hand we found the optimal a priori distribution π_0 achieving the zero rate exponent $E(+0)$ for squeezed states and computed $E(+0)$ analytically [5]. As a result it was shown that usage of squeezed states is effective in terms of the zero rate exponent. In addition we obtained the lower bound $\hat{E}_{ex}(R)$ of $E_{ex}(R)$ analytically by restricting the a priori probability distribution in the optimization to π_0, and found that squeezing is effective at low rates in terms of expurgated bound [6]. In this paper we compare the expurgated bound $E_{ex}(R)$ obtained by numerical computation with its lower bound $\hat{E}_{ex}(R)$ and see how well the latter can approximate the former.

II. Expurgated bound for squeezed state channel

The classical-quantum channel with squeezed states is defined by a channel map $\Theta : m \to \rho_m$, where ρ_m is a squeezed state given as a single-mode quantum Gaussian state with mean

$$m = \begin{pmatrix} m_1 \\ m_2 \end{pmatrix}$$

and the fixed correlation matrix

$$\alpha = \alpha(\gamma) = \hbar \begin{pmatrix} e^{-2\gamma}/2 & 0 \\ 0 & e^{2\gamma}/2 \end{pmatrix} \tag{1}$$

The energy function of the quantum Gaussian state is given by

$$f(m) = \frac{1}{2}\left(m_1^2 + m_2^2\right). \tag{2}$$

Note that we impose an energy constraint

$$\sum_{k=1}^{n} f(m(j)_k) \le \hbar n E,$$

when mapping a message j to a codeword $(m(j)_1, \ldots, m(j)_n)$. Then the expurgated bound is given [2] by

$$E_{ex}(R) = \max_{1\le s}(\max_{0\le p} \max_{\pi\in\mathcal{P}_1} \tilde{\mu}(\pi, s, p) - sR), \tag{3}$$

where $\mathcal{P}_1$ is the set of probability distribution π satisfying

$$\int f(m)\pi(dm) \le \hbar E, \tag{4}$$

and

$$\tilde{\mu}(\pi, s, p) = -s \log \int\int \mathrm{e}^{p[f(m)+f(n)-2\hbar E]} \left(\mathrm{Tr}\sqrt{\rho_m}\sqrt{\rho_n}\right)^{\frac{1}{s}} \pi(dm)\pi(dn) \tag{5}$$

is called a Gallager function. The expurgated bound can be obtained analytically for coherent states ($\gamma = 0$) as follows:
when $R < \log\vartheta(2E)$,

$$E_{ex}(R) = 2E(1 - \sqrt{1 - e^{-R}}), \tag{6}$$

and otherwise

$$E_{ex}(R) = 2E + 2 - 2\vartheta(2E) + \log\vartheta(2E) - R \quad (7)$$

with

$$\vartheta(x) = \frac{1+\sqrt{x^2+1}}{2}.$$

III. Gallager function for squeezed state channel

We compute Eq. (5) for the classical-quantum channel with squeezed states. Like the classical case, we assume the a priori distribution is Gaussian

$$\pi(dm) = \frac{1}{2\pi\sqrt{\det\beta}}\exp\left[-\frac{1}{2}m^T\beta^{-1}m\right]dm, \quad (8)$$

with

$$\beta = \hbar\begin{pmatrix} E_1 & 0 \\ 0 & E_2 \end{pmatrix}, \quad (9)$$

satisfying $E = (E_1 + E_2)/2$. Then we can restrict the region of p to make optimization easier. To see this, let us briefly review the calculations in the paper [5], where explanation of this restriction is omitted. Further calculation of Eq. (5) using the formula for the Gaussian state yields the following [5]:

$$\tilde{\mu}(\pi, s, p) = -s\log\int\int e^{-2p\hbar E}\frac{1}{4\pi^2\det\beta}\cdot \exp\left[-\frac{1}{2}\begin{pmatrix} m \\ n \end{pmatrix}^T\mathcal{A}\begin{pmatrix} m \\ n \end{pmatrix}\right]dmdn, \quad (10)$$

where

$$\mathcal{A} = \begin{pmatrix} A & B \\ B & A \end{pmatrix}, \quad (11)$$

with

$$\begin{aligned} A &= (2s\mathcal{G}_{1/2}(\alpha)\alpha)^{-1} - pI_2 + \beta^{-1}, \\ B &= -(2s\mathcal{G}_{1/2}(\alpha)\alpha)^{-1}. \end{aligned} \quad (12)$$

Here $\mathcal{G}_{1/2}(\alpha)$ takes a simple form

$$\mathcal{G}_{1/2}(\alpha) = g_{1/2}\left(\sqrt{\det\alpha}/\hbar\right)I_2$$

with the 2×2 identity matrix I_2 and

$$g_s(d) = \frac{1}{2d}\frac{(d+1/2)^s + (d-1/2)^s}{(d+1/2)^s - (d-1/2)^s}$$

as we consider the single mode case now. In Eq. (10), we can assume the matrix $\mathcal{A}$ is positive definite and hence all its eigenvalues are positive, because if this condition is not satisfied, the integral in Eq. (10) diverges to infinity and the value of $\tilde{\mu}(\pi, s, p)$ becomes $-\infty$, which will not lead to the optimal solution. Since

$$\begin{aligned} \det(\lambda I_4 - \mathcal{A}) &= \det\begin{pmatrix} \lambda I_2 - A & -B \\ -B & \lambda I_2 - A \end{pmatrix} \\ &= \det(\lambda I_2 - A - B)\det(\lambda I_2 - A + B), \end{aligned} \quad (13)$$

we find the eigenvalues of $\mathcal{A}$ are given by

$$\begin{aligned} \lambda_1 &= -p + \frac{1}{\hbar E_1}, \lambda_2 = -p + \frac{1}{\hbar E_1} + \frac{1}{sg_1\hbar} \\ \lambda_3 &= -p + \frac{1}{\hbar E_2}, \lambda_4 = -p + \frac{1}{\hbar E_2} + \frac{1}{sg_2\hbar}, \end{aligned} \quad (14)$$

with $g_1 = e^{-2\gamma}/2$ and $g_2 = e^{2\gamma}/2$. Since $\lambda_i > 0$ ($i = 1, 2, 3, 4$), it is found that we may restrict the region of parameter p in Eq.(3) as

$$0 \le p < \frac{1}{\hbar E_1}, \quad 0 \le p < \frac{1}{\hbar E_2}. \quad (15)$$

Under these constraints we can compute Eq. (10) as

$$\tilde{\mu}(\pi, s, p) = 2ps\hbar E + \frac{s}{2}\log\det\beta^2\det\mathcal{A}. \quad (16)$$

Substituting Eq. (11), we obtain

$$\tilde{\mu}(\pi, s, p) = 2ps\hbar E + \frac{s}{2}\log\det\left[(I_2 - p\beta)(I_2 - p\beta + (s\mathcal{G}_{1/2}(\alpha)\alpha)^{-1}\beta)\right], \quad (17)$$

which can be written as

$$\tilde{\mu}(\pi, s, p) = 2ps\hbar E + \frac{s}{2}\log\prod_{i=1}^{2}(1 - p\hbar E_i)(1 - p\hbar E_i + \frac{E_i}{sg_i}), \quad (18)$$

using Eqs. (1) and (9).

IV. Computation of Expurgated bound for Squeezed State Channel

We compute the expurgated bound (3) for squeezed states using Eq. (18). It is difficult to solve the optimization in Eq. (3) analytically. However, if we give up on optimizing with respect to the probability distribution π and fix it to the Gaussian one π_0 given by Eq. (8) with $E_1 = 2E$ and $E_2 = 0$, the optimization with respect to s and p can be done analytically, and we have a lower bound, $\hat{E}_{ex}(R)$, of $E_{ex}(R)$ as follows [6]:
when $R < \frac{1}{2}\log\vartheta(4Ee^{2\gamma})$,

$$\hat{E}_{ex}(R) = 2Ee^{2\gamma}(2 - \sqrt{1 - e^{-2R}}) \quad (19)$$

and otherwise

$$\hat{E}_{ex}(R) = 2Ee^{2\gamma} + 1 - \vartheta(Ee^{2\gamma}) + \frac{1}{2}\log\vartheta(4Ee^{2\gamma}) - R. \quad (20)$$

Now let us compute the expurgated bound $E_{ex}(R)$ for squeezed states numerically and compare it with $\hat{E}_{ex}(R)$. Note that we may replace the parameter p by $p/\hbar$ without loss of generality and then we can remove $\hbar$ from Eqs. (15) and (18). In the following we consider not only the signal energy E but also the squeezing energy

$$E_\gamma = \frac{\hbar}{2}\mathrm{Sp}\alpha(\gamma) - \frac{\hbar}{2}, \quad (21)$$

and we fix the total energy $E_t = E + E_\gamma$. Figure 1 represents lower bounds of the reliability function when $E_t = 1$. Here the solid line and the dotted line represent the graphs of $\hat{E}_{ex}(R)$, $E_{ex}(R)$ for squeezed states with $\gamma = 0.2$ respectively, and the dashed line that of $E_{ex}(R)$ for coherent states ($\gamma = 0$). When the value of R is small, the optimal proba-

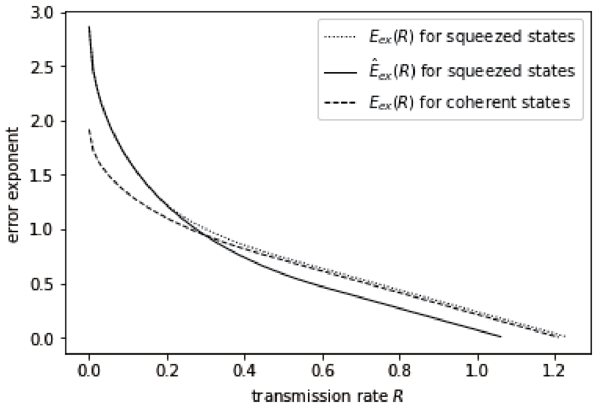

Fig. 1. Comparison of the expurgated bound $E_{ex}(R)$ for squeezed states, its lower bound $\hat{E}_{ex}(R)$ and the expurgated bound $E_{ex}(R)$ for coherent states.

bility distribution with the correlation matrix (9) for squeezed states is given by $E_1 = 2E, E_2 = 0$ and graphs of $E_{ex}(R)$ and $\hat{E}_{ex}(R)$ coincide. As R becomes somewhat larger, $\hat{E}_{ex}(R)$ no longer approximates $E_{ex}(R)$ well. Figure 2 represents the relationship between the parameter γ and the lower bounds of reliability function $E_{ex}(R)$ and $\hat{E}_{ex}(R)$ when $R = 0.2$ and $R = 0.7$. In Figure 2 (a) the dotted line represents $E_{ex}(0.2)$ and the solid line $\hat{E}_{ex}(0.2)$. In Figure 2 (b) the dotted line represents $E_{ex}(0.7)$ and the solid line $\hat{E}_{ex}(0.7)$. Both figures show that $\hat{E}_{ex}(R)$ does not give a good approximation of $E_{ex}(R)$ as a squeezing parameter γ becomes smaller.

V. Conclusion

The expurgated bound is calculated for squeezed states and compared with its lower bound $\hat{E}_{ex}(R)$ for which the analytical solution is known. However, this evaluation experiment is limited to the case of $E_t = 1$. A more comprehensive report of the experiment will be given in the future work.

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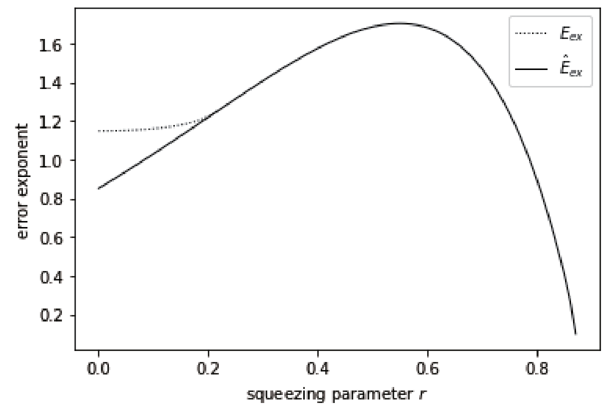
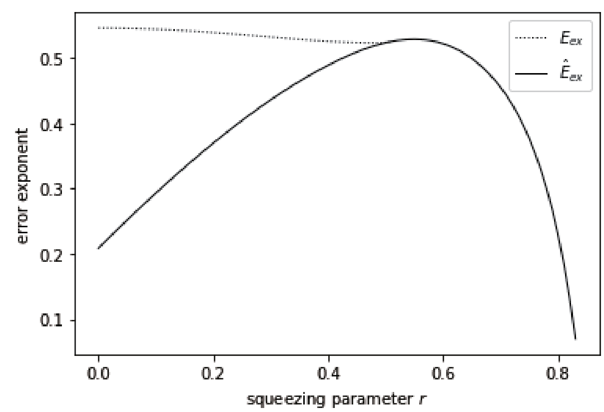


(a)



(b)

Fig. 2. Comparison of the expurgated bound $E_{ex}(R)$ and its lower bound $\hat{E}_{ex}(R)$, when (a) $R = 0.2$, (b) $R = 0.7$.

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