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Quantum Illumination with Bright Two-Mode Squeezed States

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***Abstract*—We evaluate a quantum illumination system with bright two-mode squeezed states. In order to evaluate the system, we compute a fidelity F between two two-mode Gaussian states, which is closely related to error probability of the system. And we investigate the behavior of F when a squeezing parameter r changes satisfying the constraint on sum of signal and squeezing energies. We could find out an optimum values of r for some cases by computation.**

I. Introduction

We study about quantum illumination with bright two-mode squeezed states. Tan et al. proposed quantum illumination using two-mode squeezed states with zero mean and computed its error probability [1]. Then they found that the quantum illumination system using two-mode squeezed states with zero mean has a lower value of the error probability than the system using coherent states. The two-mode squeezed state is a Gaussian state which is characterized by a mean vector and a correlation matrix, and has signal energy N_s corresponding to the mean vector and squeezing energy N_{sq} corresponding to the correlation matrix. On the other hand the coherent state is a Gaussian state having only signal energy N_s , i.e. $N_{sq} = 0$. With such terms, it can be said they compared effects of two different Gaussian states, one with $N_{sq} > 0$ and $N_s = 0$ and one with $N_{sq} = 0$ and $N_s > 0$ under the condition that $N_{sq} + N_s$ is a constant. The main objective of this paper is to confirm how effective it is to use bright two-mode squeezed states, which are two-mode Gaussian states with $N_{sq} > 0$ and $N_s > 0$. Note that we consider an asymmetric case where a mean vector for an idler mode is set to zero and only signal mode has a non-zero mean vector. In order to evaluate the quantum illumination system, we use a fidelity, which is strongly related to the error probability of the system.

II. Two-mode Gaussian states

A. Preliminaries

Let us recall a theory of general Gaussian state [4]. We consider a quantum system, such as a cavity field with finite numbers of modes, described by the annihilation operators $a_1, \ldots, a_n$ satisfying the canonical commutation relation (CCR)

$$[a_j, a_k^\dagger] = \delta_{j,k} I, \qquad [a_j, a_k] = 0, \tag{1}$$

where I is the unit operator, $\dagger$ denotes the adjoint operation and

$$\delta_{j,k} = \begin{cases} 1 & j = k \\ 0 & j \neq k \end{cases} . \tag{2}$$

The Hilbert space of irreducible representation of this CCR is denoted by $\mathcal{H}$. Let us introduce canonical pairs

$$q_j = \sqrt{\frac{\hbar}{2}}(a_j + a_j^\dagger), \quad p_j = i\sqrt{\frac{\hbar}{2}}(a_j^\dagger - a_j), \tag{3}$$

such that

$$a_j = \frac{1}{\sqrt{2\hbar}}(\ q_j + ip_j), \tag{4}$$

satisfying the Heisenberg CCR

$$[q_j, p_k] = i\delta_{j,k}\hbar I, \quad [q_j, q_k] = 0, \quad [p_j, p_k] = 0. \tag{5}$$

For simplicity we use a column vector

$$\mathcal{R} = [q_1, p_1; \ldots; q_n, p_n]^T, \tag{6}$$

and a real column $2n-$vector

$$z = [x_1, y_1; \ldots; y_n, y_n]^T \tag{7}$$

in the following. Let us introduce a unitary operator in $\mathcal{H}$

$$V(z) = \exp i \sum_{j=1}^{n}(x_j q_j + y_j p_j) = \exp(i\mathcal{R}^T z). \tag{8}$$

The operator $V(z)$ satisfies the Weyl-Segal CCR

$$V(z)V(z') = \exp[\frac{i}{2}\Delta(z, z')]V(z + z'), \tag{9}$$

where

$$\Delta(z, z') = \hbar \sum_{j=1}^{n}(x'_j y_j - x_j y'_j) \tag{10}$$

is a canonical symplectic form. Let us consider $2n \times 2n-$skew-symmetric commutation matrix Δ_n of components of the vector $\mathcal{R}$, which is given as

$$\Delta_1 = \begin{bmatrix} 0 & \hbar \\ -\hbar & 0 \end{bmatrix} \tag{11}$$

for $n = 1$, and

$$\Delta_n = \begin{bmatrix} \Delta_1 & & & \\ & \Delta_1 & & \\ & & \ddots & \\ & & & \Delta_1 \end{bmatrix} \tag{12}$$

for $n > 1$. Then the canonical symplectic form (10) can be written as

$$\Delta(z, z') = -z^T \Delta_n z'. \tag{13}$$

Let us mention that if $\zeta = (\zeta_1, \ldots, \zeta_n)$ where $\zeta_j = \frac{1}{\sqrt{2\hbar}}(x_j + iy_j)$, then the displacement operator

$$D_n(z) = \exp \sum_{j=1}^{n} (\zeta_j a_j^\dagger - \bar{\zeta}_j a_j) \tag{14}$$

is written by $V(z)$ as

$$D_n(z) = \exp \frac{i}{\hbar} \sum_{j=1}^{n} (y_j q_j - x_j p_j) = V(-\Delta_n^{-1} z). \tag{15}$$

The quantum characteristic function is defined to be

$$\mathrm{Tr}\rho V(z) \tag{16}$$

The density operator is called Gaussian, if its quantum characteristic function has the form

$$\mathrm{Tr}\rho V(z) = \exp(im^T z - \frac{1}{2} z^T \alpha z), \tag{17}$$

where m is a column $2n-$vector and α is a real symplectic $2n \times 2n$-matrix. One can show that

$$m = \mathrm{Tr}\rho\mathcal{R}; \alpha - \frac{i}{2}\Delta_n = \mathrm{Tr}\mathcal{R}\rho\mathcal{R}^T. \tag{18}$$

The mean m can be arbitrary vector; the necessary and sufficient condition on the correlation matrix α is the matrix uncertainty relation

$$\alpha - \frac{i}{2}\Delta_n \geq 0. \tag{19}$$

B. Two-mode squeezed states

Let a_S and a_I be annihilation operators for the entangled signal and idler mode pair. We neglect an unimportant phase factor, and the two-mode squeezed state can be written as

$$|\psi_{sq}\rangle_{SI} = S(r)|0\rangle_{SI}. \tag{20}$$

Here, $|0\rangle_{SI}$ is the two-mode vacuum state $|0\rangle_S \otimes |0\rangle_I$, and

$$S(r) = \exp[-r(a_S^\dagger a_I^\dagger - a_S a_I)] \tag{21}$$

is a squeezing operator, which transforms annihilation operators a_S and a_I as

$$a'_S = S(r)^\dagger a_S S(r) = a_S \cosh r - a_I^\dagger \sinh r \tag{22}$$

$$a'_I = S(r)^\dagger a_I S(r) = -a_S^\dagger \sinh r + a_I \cosh r. \tag{23}$$

We introduce a vector representation $\mathcal{R} = [q_S, p_S; q_I, p_I]^T$ and define a unitary operator $V(z)$ for a 4-dimension real vector z as in Section II-A. Using the vector representations $\mathcal{R}$ and $\mathcal{R}' = [q'_S, p'_S; q'_I, p'_I]$ with $q'_j = S(r)^\dagger q_j S(r)$ and $p'_j = S(r)^\dagger p_j S(r)$ $(j = S, I)$, we can rewrite Eqs. (22) and (23) in a real setting as

$$\mathcal{R}' = \mathcal{L}\mathcal{R} \tag{24}$$

Here $\mathcal{L}$ is written as

$$\mathcal{L} = \begin{pmatrix} \cosh r I_2 & \sinh r J_2 \\ \sinh r J_2 & \cosh r I_2 \end{pmatrix} \tag{25}$$

with

$$I_2 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \; J_2 = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}. \tag{26}$$

From Eq. (24), we can obtain

$$\begin{aligned} S(r)^\dagger V(z) S(r) &= \exp[i(\mathcal{R}')^T z] \\ &= \exp[i\mathcal{R}^T(\mathcal{L}^T z)] \\ &= V(\mathcal{L}^T z). \end{aligned} \tag{27}$$

Thus, the characteristic function of $|\psi_{sq}\rangle_{SI}$ is given as

$$\begin{aligned} \mathrm{Tr}|\psi_{sq}\rangle_{SI}\langle\psi_{sq}|V(z) &= \mathrm{Tr}|0\rangle_{SI}\langle 0|S(r)^\dagger V(z) S(r) \\ &= \exp\left[-\frac{1}{2}\frac{\hbar}{2} z^T \mathcal{L}\mathcal{L}^T z\right], \end{aligned} \tag{28}$$

and the correlation matrix of $|\psi_{sq}\rangle_{SI}$ can be written as

$$\alpha = \frac{\hbar}{2}\mathcal{L}\mathcal{L}^T = \frac{\hbar}{2} \begin{pmatrix} \cosh 2r I_2 & \sinh 2r J_2 \\ \sinh 2r J_2 & \cosh 2r I_2 \end{pmatrix}. \tag{29}$$

III. Quantum illumination system with Gaussian states

We kindly explain our setting of quantum illumination system with Gaussian states. The system generates a bright two-mode squeezed state $\rho_{SI}(z) = [D_1(z) \otimes I_I] |\psi_{sq}\rangle_{SI}\langle\psi_{sq}| [D_1(z)^\dagger \otimes I_I]$ and transmits its signal mode and retains the idler mode. We assume we receive the state $\rho_{SI}^{(1)} = (\Phi_k \otimes I_I)\, \rho_{SI}(z)$ with the lossy bosonic channel of transmittance k if a target exists, and we have the state $\rho_{SI}^{(0)} = |0\rangle_S\langle 0| \otimes \mathrm{Tr}_S \rho_{SI}$ if no target exists. Then we may detect which of the signals $\rho_{SI}^{(0)}$ and $\rho_{SI}^{(1)}$ is received.

The state $\rho_{SI}^{(0)}$ is a Gaussian state with the zero mean vector and the correlation matrix

$$\alpha^{(0)} = \frac{\hbar}{2} \left[\begin{array}{cc} I_2 & 0 \\ 0 & \cosh 2r I_2 \end{array} \right]. \tag{30}$$

The mean vector of the Gaussian states $\rho_{SI}(z)$ is $(x_1, y_1, 0, 0)^T$, and its correlation matrix is given by Eq. (29). Then the mean vector of the Gaussian state $\rho_{SI}^{(1)}$ is $u = (kx_1, ky_1, 0, 0)^T$. In order to obtain the correlation matrix of $\rho_{SI}^{(1)}$, we describe the transformation $|0\rangle_{SI}\langle 0| \to (\Phi_k \otimes I_I)\rho_{SI}(z)$ in the Heisenberg picture by the relation

$$\begin{aligned} \tilde{a}_S &= k a_S \cosh r - k a_I^\dagger \sinh r + \sqrt{1-k^2} a_E \\ \tilde{a}_I &= -a_S^\dagger \sinh r + a_I \cosh r, \end{aligned} \tag{31}$$

where a_E is an annihilation operator in a mode of environment. We put $\tilde{q}_j = \sqrt{\hbar/2}(\tilde{a}_j + \tilde{a}_j^\dagger), \tilde{p}_j = i\sqrt{\hbar/2}(\tilde{a}_j^\dagger - \tilde{a}_j), q_j = \sqrt{\hbar/2}(a_j + a_j^\dagger), p_j = i\sqrt{\hbar/2}(a_j^\dagger - a_j)$ with $j = S, I, E$ and we introduce the vector representation $\tilde{\mathcal{R}} = [\tilde{q}_S, \tilde{p}_S, \tilde{q}_I, \tilde{p}_I], \mathcal{R}_0 = [q_S, p_S, q_I, p_I, q_E, p_E]^T$. We can rewrite Eq. (31) in a real setting as

$$\tilde{\mathcal{R}} = \mathcal{M}\mathcal{R}_0, \tag{32}$$

where

$$\mathcal{M} = \left(\begin{array}{ccc} k \cosh r I_2 & k \sinh r J_2 & \sqrt{1-k^2} I_2 \\ \sinh r J_2 & \cosh r I_2 & 0 \end{array} \right). \tag{33}$$

As in Section II-B, we can find the correlation matrix of $\rho_{SI}^{(1)}$ as

$$\begin{aligned} \alpha^{(1)} &= \frac{\hbar}{2} \mathcal{M}\mathcal{M}^T \\ &= \frac{\hbar}{2} \left[\begin{array}{cc} (k^2 \cosh 2r + 1 - k^2) I_2 & k \sinh 2r J_2 \\ k \sinh 2r J_2 & \cosh 2r I_2 \end{array} \right]. \end{aligned} \tag{34}$$

Moreover, in the Gaussian case, we can separately deal with the effect of attenuation or amplification and that of thermal noise. Hence, when we also assume the effect of thermal noise with zero mean and variance $\hbar N_c$, we may consider the following correlation matrices as $\alpha^{(0)}$ and $\alpha^{(1)}$ respectively,

$$\begin{aligned} &\alpha^{(0)} + \hbar \left[\begin{array}{cc} N_c I_2 & 0 \\ 0 & 0 \end{array} \right], \\ &\alpha^{(1)} + \hbar \left[\begin{array}{cc} N_c I_2 & 0 \\ 0 & 0 \end{array} \right]. \end{aligned} \tag{35}$$

We constrain sum of the signal energy $N_s = x_1^2 + y_1^2$ and the squeezing energy $N_{sq} = (\cosh 2r - 1)/2$ as

$$N_s + N_{sq} = N. \tag{36}$$

In the following, we investigate the behavior of fidelity

$$F = F(\rho_{SI}^{(0)}, \rho_{SI}^{(1)}) = \mathrm{Tr} \left| \sqrt{\rho_{SI}^{(0)}} \sqrt{\rho_{SI}^{(1)}} \right| \tag{37}$$

under the energy constraint (36). We can not evaluate the quantum illumination system rigorously by the fidelity, but it is related to the error probability of the system as follows. We can evaluate the quantum illumination system by error probability of quantum detection for two equiprobable states $\rho_{SI}^{(0)}$ and $\rho_{SI}^{(1)}$. Then the minimum error probability is given as

$$P = \frac{1}{2} \min_{\{X_0, X_1\}} \left[\mathrm{Tr} X_0 \rho_{SI}^{(1)} + \mathrm{Tr} X_1 \rho_{SI}^{(0)} \right], \tag{38}$$

where the minimization is taken over all POVMs $\{X_0, X_1\}$, which consist of positive operators satisfying $X_0 + X_1 = I$. It is known that the minimum error probability P is obtained as

$$P = \frac{1}{2}(1 - D(\rho_{SI}^{(0)}, \rho_{SI}^{(1)})), \tag{39}$$

where $D(\rho_{SI}^{(0)}, \rho_{SI}^{(1)}) = \frac{1}{2}\mathrm{Tr}|\rho_{SI}^{(0)} - \rho_{SI}^{(1)}|$ is a trace distance. The trace distance is bounded above and below using the fidelity F as

$$1 - F \le D(\rho_{SI}^{(0)}, \rho_{SI}^{(1)}) \le \sqrt{1 - F^2}, \tag{40}$$

and hence we got

$$\frac{1}{2}(1 - \sqrt{1 - F^2}) \le P \le \frac{F}{2}. \tag{41}$$

Note that we may repeat the transmission described above M times to improve performance of the system. Then the minimum error probability $P^{(M)}$ is bounded as

$$P^{(M)} \le P_{QC}^{(M)} \le \frac{1}{2} F^M, \tag{42}$$

where $P_{QC}^{(M)} = \frac{1}{2} \left[\min_{0 \le s \le 1} \mathrm{Tr} (\rho_{SI}^{(0)})^s (\rho_{SI}^{(1)})^{1-s} \right]^M$ is the quantum Chernoff bound [1],[3].

IV. Computation of fidelity under total energy constraint

We can compute the fidelity for the Gaussian states $\rho_{SI}^{(0)}$ and $\rho_{SI}^{(1)}$ using the formula in [2] as

$$F(\rho_{SI}^{(0)}, \rho_{SI}^{(1)}) = \exp\left[-\frac{1}{4}u^T(\alpha^{(0)}+\alpha^{(1)})^{-1}u\right] \times \left[\left(\sqrt{\Gamma}+\sqrt{\Lambda}\right) - \sqrt{\left(\sqrt{\Gamma}+\sqrt{\Lambda}\right)^2 - \Upsilon}\right]^{-1/2} \quad (43)$$

where

$$\Upsilon := \det(\alpha^{(0)}+\alpha^{(1)}) \geq 1, \quad (44)$$

$$\Gamma := 16\det\left[(J_2\alpha^{(0)})(J_2\alpha^{(1)}) - \frac{1}{4}I\right] \geq \Upsilon, \quad (45)$$

$$\Lambda := 16\det\left(\alpha^{(0)}+\frac{i}{2}J_2\right)\det\left(\alpha^{(1)}+\frac{i}{2}J_2\right) \geq 0 \quad (46)$$

Note that when we consider effect of thermal noise we may use the correlation matrices given by Eq. (35) as $\alpha^{(0)}$ and $\alpha^{(1)}$.

Figure 1 shows graphs of the first term and the second term of Eq. (43) with respect to a squeezing parameter $r \geq 0$. From the energy constraint (36), r changes in the region

$$0 \leq r \leq r_{max} := \frac{1}{2}\log(2N+1+\sqrt{4N^2+4N}). \quad (47)$$

The first term represents effect of signal energy, and its value monotonically increases with respect to r. The second term represents the effect of squeezing, and its value monotonically decreases. In this case the second term is more effective than the first term and hence the fidelity monotonically decreases, as squeezing becomes stronger.

Figure 2 shows the relationship between the fidelity F and the squeezing parameter r for $N = 0.1, 1.0, 5.0$ each with $k = 1.0$ and $N_c = 0$. Note that each region of r is determined by (47) differently. In these cases, as r is increased the fidelity is reduced, that is, squeezing is effective.

Figure 3 shows the relationship between the fidelity F and the squeezing parameter r for $N = 20.0$, $k = 1.0$, and $N_c = 1.5$. Here the value of fidelity is 0.079288 as $r = 0$ and 0.130617 as $r = r_{max}$. The minimum value of fidelity is 0.078232 which is given by $r = 0.3497$. Hence we are necessary to choose an intermediate value of squeezing parameter r to obtain the optimum result. Figure 4 shows graphs of the first term and second term of Eq. (43) under the parameter setting of Fig. 3. In this case in the regions close to both ends either the first term or the second term takes values close to zero. Therefore the fidelity takes the maximum value in middle region in Fig. 3.

Figure 5 shows the relationship between the fidelity F and the squeezing parameter r for $k = 0.5$ and $N_c = 0$, and the parameter N is taken from $0.1, 1.0, 5.0$ and 20.0. Except in the case of $N = 20.0$ the graphs have the same tendency as in the case of $k = 1$. In the case of $N = 20.0$, the graph behaves similarly to the graph of Fig. 3. This means squeezing becomes less effective than displacement as the value of N increases. On the other hand, Fig. 6 shows the graph of fidelity for $N = 5.0, 20.0, 50.0, 100.0$ each with $k = 0.5$ and $N_c = 2.0$. From this we can see that when we consider a thermal noise we need a larger value of N so that the graph behaves similarly to the graph of Fig. 3.

Figure 7 shows the relationship between fidelity and a squeezing parameter r for $N = 100$ and $N_c = 0$, and the parameter k is taken from $0.05, 0.1, 0.25$ and 0.5. From this we can see even if N takes a large value squeezing is useful for small values of transmittance k, $k = 0.05, 0.1$. On the other hand for larger values of k, $k = 0.25$ and 0.5, the graphs have the same tendency as the graph of Fig. 3.

V. Conclusion

We have computed the fidelity between two-mode Gaussian states used in the quantum illumination system under the total energy constraint $N_s + N_{sq} = N$. And we have drawn graphs of fidelity $F = F(r)$ with respect to a squeezing parameter r for various cases of k, N_c, and N. When energy of received signal is small, the minimum value of fidelity is given by $N_s = 0$ (i.e. $r = r_{max}$). Such cases are also studied in a previous research [1]. In this paper we have newly found there is a case where an intermediate value of r ($0 < r < r_{max}$) gives the minimum fidelity.

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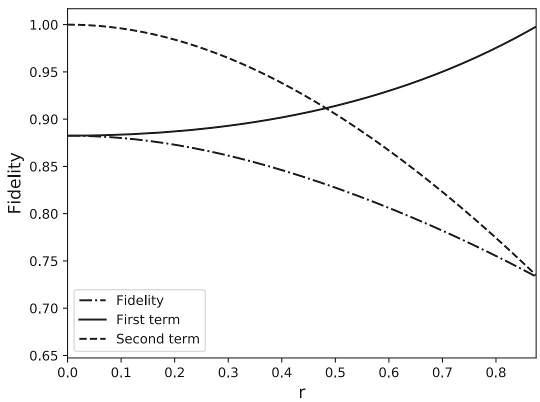

Fig. 1. Behavior of first and second terms in the fidelity formula and the fidelity for $k = 0.5, N = 1.0, N_c = 0$.

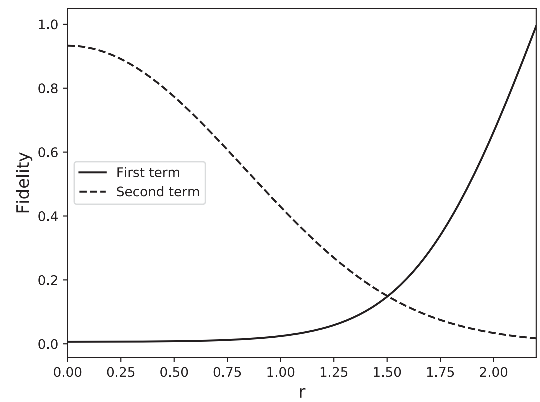

Fig. 4. Behavior of first and second terms in the fidelity formula for $k = 1.0, N = 20.0, N_c = 1.5$

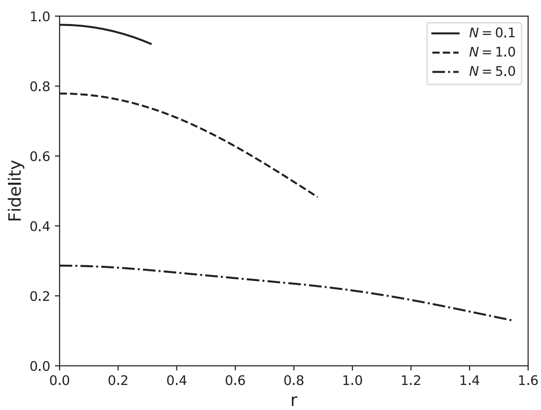

Fig. 2. Relationship between fidelity F and squeezing parameter r for $N = 0.1, 1.0, 5.0$ each with $k = 1.0$ and $N_c = 0$.

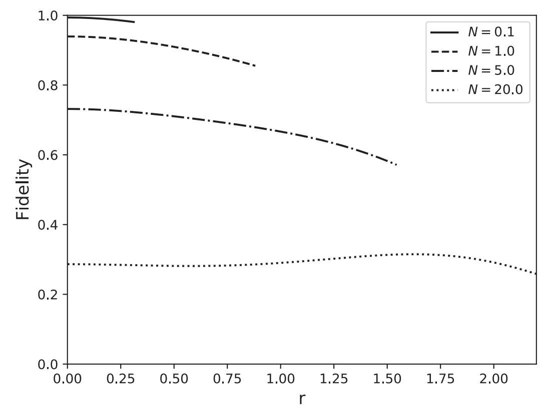

Fig. 5. Relationship between fidelity F and squeezing parameter r for $N = 0.1, 1.0, 5.0, 20.0$ each with $k = 0.5$ and $N_c = 0$.

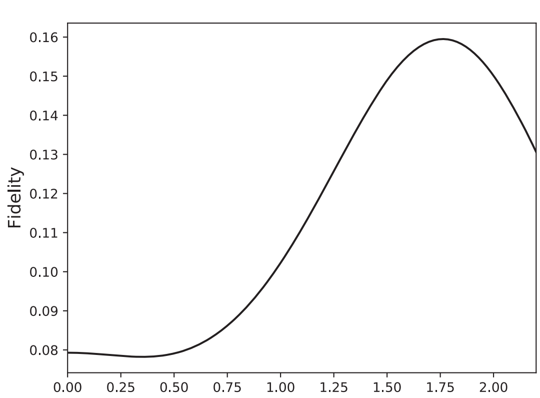

Fig. 3. Relationship between fidelity F and squeezing parameter r for $k = 1.0$, $N = 20.0$ and $N_c = 1.5$.

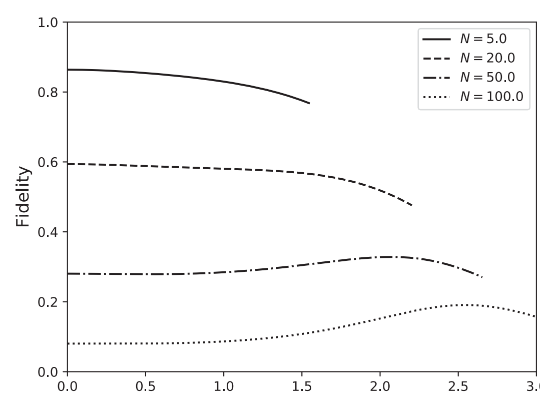

Fig. 6. Relationship between fidelity F and squeezing parameter r for $N = 5.0, 20.0, 50.0, 100.0$ each with $k = 0.5$ and $N_c = 2.0$.

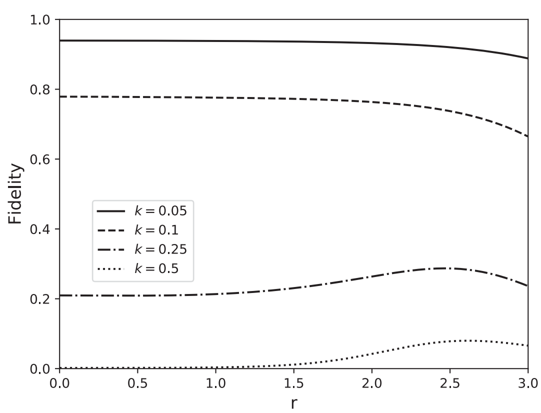

Fig. 7. Relationship between fidelity F and squeezing parameter r for $k = 0.05, 0.1, 0.25, 0.5$ each with $N = 100$ and $N_c = 0$.