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Optimal discrimination of symmetric quantum pure states except one and its application to near-optimal optical receivers

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***Abstract*—We derive an analytical solution to a minimum-error measurement for quantum pure states with equal real inner products whose prior probabilities are equal except for one. Pulse position modulated (PPM) optical coherent signals are one important example of these states. We also show that a Dolinar-like receiver, which is based on feedback control, for PPM coherent signals achieves near-optimal performance.**

I. Introduction

Quantum state discrimination is a fundamental and difficult problem in quantum information and quantum communication. The problem is to discriminate between the quantum state from a given finite set of known possible states with given prior probabilities. A well-known feature of quantum mechanics is that it is impossible to perfectly discriminate between nonorthogonal quantum states. Thus, an optimum quantum measurement must be found that can discriminate between them as accurately as possible.

The problem of finding an optimal quantum measurement for distinguishing between quantum states was first formulated in the 1970s by Helstrom, Holevo, and Yuen *et al.* [1]–[3]. Holevo and Yuen *et al.* [2], [3] formulated necessary and sufficient conditions for obtaining an optimal measurement that minimizes the average probability of a detection error, which is called a minimum-error measurement. However, obtaining a closed-form analytical expression for a minimum-error measurement is generally a very difficult task. In recent years, many research studies have been conducted to analytically obtain a minimum-error measurement for some states. In the case in which states have some kind of symmetry and equal prior probabilities, closed-form analytical expressions for minimum-error measurements have been derived, for example, with cyclic pure states [4], [5], ternary mirror-symmetric states [6], linear codes with binary letter-states [7], pseudo-cyclic codes with q-ary letter-states [8], and geometrically uniform pure states [9], [10]. Note that, analytical solutions of a minimum-error measurement for mixed geometrically uniform states have also been derived [10]–[14].

Unfortunately, optimal measurements are often practically difficult to implement. Thus, how to construct a quantum receiver that can discriminate between the given states as accurately as possible is a crucial issue. Dolinar [15] proposed a receiver based on a combination of a beam splitter, a local coherent light source, a photon detector, and a feedback circuit, which achieves a minimum-error measurement for binary optical coherent states. In this receiver, the posterior probabilities are updated in each feedback period, and a minimum-error measurement is performed during the subsequent period. The symmetry that given binary coherent states have is broken since updated probabilities are different. Dolinar-like protocols might be useful for more than two states, which means that a Dolinar-like receiver for the states might achieve near-optimal performance. However, an analytical solution to a minimum-error measurement is not known when their probabilities are different, even if the states have symmetry. We want to find a minimum-error measurement to implement an optimal or near-optimal receiver. In this paper, we show that there is a case in which a minimum-error measurement for more than two quantum states with different proir probabilities can be analytically derived. We focus on quantum pure states with equal real inner products. These states can represent pulse position modulated (PPM) optical coherent signals, which are currently widely adopted in deep-space optical communications. We also show that a Dolinar-like receiver for PPM coherent signals achieves near-optimal performance and that this receiver can be applied to other optical signals.

II. Quantum pure states with equal real inner products

Suppose that $\{|\Psi'_m\rangle\}_{m\in\mathcal{I}_M}$ are M quantum pure states with equal real inner products, where $\mathcal{I}_M = \{0, 1, \cdots, M-1\}$. Each vector $|\Psi'_m\rangle$ is normalized. Since the binary case (i.e., $M = 2$) is trivial, we assume $M \geq 3$. The inner product between $|\Psi'_j\rangle$ and $|\Psi'_k\rangle$ is written by

$$\langle\Psi'_j|\Psi'_k\rangle = \begin{cases} 1, & j = k, \\ K, & \text{otherwise}, \end{cases} \quad (1)$$

where $K < 1$ is real. Let us represent each $|\Psi'_m\rangle$ with respect to an orthonormal basis $\{|\phi_n\rangle\}_{n\in\mathcal{I}_M}$. Let Ψ be the M-dimensional square matrix whose $(m+1)$-th colomn is $|\Psi'_m\rangle = [\langle\phi_0|\Psi'_m\rangle, \cdots, \langle\phi_{M-1}|\Psi'_m\rangle]^T$ (^{T} is the transpose operator), i.e.,

$$\Psi \quad := \quad \begin{bmatrix} \langle\phi_0|\Psi'_0\rangle & \cdots & \langle\phi_0|\Psi'_{M-1}\rangle \\ \vdots & \ddots & \vdots \\ \langle\phi_{M-1}|\Psi'_0\rangle & \cdots & \langle\phi_{M-1}|\Psi'_{M-1}\rangle \end{bmatrix}. \qquad (2)$$

The eigenvalues of $\Psi^\dagger\Psi$ († is the conjugate transpose operator) are $1-K$ and $1+(M-1)K$. Since $\Psi^\dagger\Psi$ is positive semidefinite, $1+(M-1)K \ge 0$, i.e., $K \ge K_{\min} = -1/(M-1)$, must hold. rank $\Psi = M$ holds if $K > K_{\min}$; otherwise (i.e., $K = K_{\min}$), rank $\Psi = M-1$ holds.

By choosing an appropriate orthonormal basis $\{|\phi_n\rangle\}$, the pure state $|\Psi'_m\rangle$ can be expressed as

$$|\Psi'_m\rangle \quad = \quad b\sum_{n=0}^{M-1}|\phi_n\rangle + (a-b)\,|\phi_m\rangle, \qquad (3)$$

where

$$\begin{aligned} b \quad &:= \quad \frac{\sqrt{1-K+MK} - \sqrt{1-K}}{M}, \\ a \quad &:= \quad \frac{\sqrt{1-K+MK} + (M-1)\sqrt{1-K}}{M}. \end{aligned} \qquad (4)$$

Indeed, the state $|\Psi'_m\rangle$ of Eq. (3) satisfies Eq. (1). Also, if the states $\{|\Psi'_m\rangle\}$ are not expressed in the form of Eq. (3), then, by changing the basis $\{|\phi_n\rangle\}$, we can rewrite in the form of Eq. (3) (see Appendix A). Note that, although the space spanned by $\{|\Psi'_m\rangle\}$ is $(M-1)$-dimensional when $K = K_{\min}$, we always consider the M-dimensional complex Hilbert space spanned by $\{|\phi_m\rangle\}$.

Let $|\Psi_m\rangle := \sqrt{\xi_m}\,|\Psi'_m\rangle$, where ξ_m is the prior probability of $|\Psi'_m\rangle$. Let us consider a finite group $\mathcal{G}$ whose every element is a unitary operator. The pure states $\{|\Psi_m\rangle\}_{m\in\mathcal{I}_M}$ are group covariant with respect to $\mathcal{G}$, which we refer to as $\mathcal{G}$-symmetric states, if, for any $m \in \mathcal{I}_M$ and $U \in \mathcal{G}$, there exists $k \in \mathcal{I}_M$ such that $|\Psi_k\rangle = U\,|\Psi_m\rangle$ [16]. Here, we consider the symmetric group on $\mathcal{I}_M$, $S(\mathcal{I}_M)$, which contains all the permutations on $\mathcal{I}_M$. For any permutation $g \in S(\mathcal{I}_M)$, which transforms $m \in \mathcal{I}_M$ into $g(m) \in \mathcal{I}_M$, the unitary operator U_g is defined as

$$U_g \quad := \quad \sum_{m=0}^{M-1}|\phi_m\rangle\langle\phi_{g(m)}|. \qquad (5)$$

It is easily seen that $U_g\,|\Psi_m\rangle = |\Psi_{g(m)}\rangle$ holds for any $m \in \mathcal{I}_M$. Thus, let $\mathcal{G}_0$ be the group containing all the permutations $g \in S(\mathcal{I}_M)$ satisfying $\xi_{g(m)} = \xi_m$ for any $m \in \mathcal{I}_M$ and let $\mathcal{G} := \{U_g : g \in \mathcal{G}_0\}$; then, $\{|\Psi_m\rangle\}_{m\in\mathcal{I}_M}$ is $\mathcal{G}$-symmetric. For example, if $\xi_1 = \xi_2 = \cdots = \xi_{M-1}$, then $\{|\Psi_m\rangle\}_{m\in\mathcal{I}_M}$ is $S(\mathcal{I}_M\backslash\{0\})$-symmetric, where '\' is the set difference operator.

III. Optimal discrimination of quantum pure states with equal real inner products

We will derive a minimum-error measurement for pure states with equal real inner products. Assume that $\xi_0 \ne \xi_1$ and $\xi_1 = \xi_2 = \cdots = \xi_{M-1}$ hold; i.e., the prior probabilities except for ξ_0 are equal. Such a case can be utilized for realization of near-optimal receiver that will be described in the next section.

We consider a quantum measurement that consists of M detection operators, which is represented by a positive operator valued measure (POVM) $\Pi := \{\Pi_m\}_{m\in\mathcal{I}_M}$. Any POVM Π must satisfy

$$\begin{aligned} \Pi_m \quad &\ge \quad 0, \quad \forall m \in \mathcal{I}_M, \\ \sum_{m=0}^{M-1}\Pi_m \quad &= \quad I, \end{aligned} \qquad (6)$$

where I is the identity matrix and $\Pi_m \ge 0$ denotes that the operator Π_m is positive semidefinite. The operator Π_m for each $m \in \mathcal{I}_M$ corresponds to detection of the state $\hat{\rho}_m$. The average success probability $P_{\rm S}$ is defined by

$$P_{\rm S}(\Pi) \quad := \quad \sum_{m=0}^{M-1}\langle\Psi_m|\Pi_m|\Psi_m\rangle. \qquad (7)$$

Also, the average success probability $P_{\rm E}$ is defined by $P_{\rm E} := 1 - P_{\rm S}$. We want to find a minimum-error measurement, which is an optimal solution to the following optimization problem:

$$\begin{aligned} &\text{maximize} \quad P_{\rm S}(\Pi) \\ &\text{subject to} \quad \Pi \text{ is a POVM.} \end{aligned} \qquad (8)$$

Let $\mathcal{H}$ be the M-dimensional complex Hilbert space spanned by $\{|\phi_m\rangle\}$. In the case of $K > K_{\min}$, the states $\{|\Psi'_m\rangle\}$ are linearly independent, in which case it is known that a minimum-error measurement is a von Neumann measurement [17]–[19]. Also, in the case of $K = K_{\min}$, we can easily see that the exists a von Neumann measurement that is a minimum-error measurement by considering $K = K_{\min} + \epsilon$ $(\epsilon > 0)$ in the limit $\epsilon \to 0$. A von Neumann measurement that is a minimum-error measurement can be expressed by $\Pi = \{\Pi_m = |\pi_m\rangle\langle\pi_m|\}_{m\in\mathcal{I}_M}$ with unit vectors $|\pi_m\rangle$.

Since the states $\{|\Psi_m\rangle\}$ is $S(\mathcal{I}_M\backslash\{0\})$-symmetric, there exists a minimum-error measurement $\{|\pi_m\rangle\langle\pi_m|\}_{m\in\mathcal{I}_M} \subset \mathcal{H}$ with $S(\mathcal{I}_M\backslash\{0\})$-symmetric [16], i.e., satisfying $U_g\,|\pi_m\rangle = |\pi_{g(m)}\rangle$. Let $g_{j,k}$ be the permutation of $\mathcal{I}_M$ that interchanges j and k with $j,k \in \mathcal{I}_M\backslash\{0\}$. The unitary operator $U_{g_{j,k}}$ satisfies

$$\begin{aligned} U_{g_{j,k}}\,|\pi_m\rangle \;&=\; |\pi_m\rangle, \quad m \in \mathcal{I}_M,\; j,k \in \mathcal{I}_M\backslash\{0,m\}, \\ U_{g_{m,k}}\,|\pi_m\rangle \;&=\; |\pi_k\rangle, \quad m,k \in \mathcal{I}_M\backslash\{0\}. \end{aligned} \qquad (9)$$

By substituting $|\pi_m\rangle = \sum_{n=0}^{M-1}|\phi_n\rangle\langle\phi_n|\pi_m\rangle$ we have

$$\begin{aligned} \langle\phi_j|\pi_m\rangle \;&=\; \langle\phi_k|\pi_m\rangle, \quad m \in \mathcal{I}_M,\; j,k \in \mathcal{I}_M\backslash\{0,m\}, \\ \langle\phi_j|\pi_m\rangle \;&=\; \langle\phi_j|\pi_k\rangle, \quad m,k \in \mathcal{I}_M\backslash\{0\},\; j \in \mathcal{I}_M\backslash\{m,k\}, \\ \langle\phi_m|\pi_m\rangle \;&=\; \langle\phi_k|\pi_k\rangle, \quad m,k \in \mathcal{I}_M\backslash\{0\}. \end{aligned} \qquad (10)$$

Thus, we can reduce the M^2 unknown parameters (i.e., $\langle\phi_0|\pi_0\rangle, \cdots, \langle\phi_{M-1}|\pi_{M-1}\rangle$) to five: we can write, for any distinct $j, m \in \mathcal{I}_M\backslash\{0\}$, $\langle\phi_0|\pi_0\rangle =: x_0$, $\langle\phi_j|\pi_0\rangle =: x_1$, $\langle\phi_m|\pi_m\rangle =: x_2$, $\langle\phi_j|\pi_m\rangle =: x_3$, and $\langle\phi_0|\pi_m\rangle =: x_4$. Since $\langle\phi_k|\Psi_m\rangle$ are real numbers, we can assume, without loss of generality, that $x_0, \cdots, x_4$ are also real numbers.

We now show that $x_1, \cdots, x_4$ can be written by using x_0. Let us choose the phases of $\{|\phi_m\rangle\}$ and $\{|\pi_m\rangle\}$ such that $x_0 \geq 0$ and $x_1 x_4 \geq 0$ hold. Let $\Gamma := [|\pi_0\rangle, \cdots, |\pi_{M-1}\rangle]$. From Eq. (6), Γ is unitary, i.e., $\Gamma^\dagger\Gamma = \Gamma\Gamma^\dagger = I$ holds. Thus, we have

$$\begin{aligned} x_0^2 + (M-1)x_1^2 &= 1, \\ x_0^2 + (M-1)x_4^2 &= 1, \\ x_1^2 + x_2^2 + (M-2)x_3^2 &= 1, \\ x_0 x_4 + x_1\{x_2 + (M-2)x_3\} &= 0, \\ x_1^2 + 2x_2x_3 + (M-3)x_3^2 &= 0, \end{aligned} \quad (11)$$

which yields

$$\begin{aligned} x_1 &= s_1\sqrt{\frac{1-x_0^2}{M-1}}, \\ x_2 &= s_2\frac{M-2}{M-1} - \frac{x_0}{M-1}, \\ x_3 &= -\frac{s_2 + x_0}{M-1}, \\ x_4 &= x_1, \end{aligned} \quad (12)$$

where $s_1 = \pm 1$ and $s_2 = \pm 1$.

The average success probability can be written as

$$\begin{aligned} P_\mathrm{S} &= \xi_0\{ax_0 + (M-1)bx_1\}^2 \\ &\quad + (1-\xi_0)\{bx_1 + ax_2 + (M-2)bx_3\}^2. \end{aligned} \quad (13)$$

Substituting Eq. (12) into this equation gives

$$\begin{aligned} P_\mathrm{S} &= \xi_0\left[ax_0 + bs_1\sqrt{(M-1)(1-x_0^2)}\right]^2 \\ &\quad + \frac{1-\xi_0}{(M-1)^2}\left[bs_1\sqrt{(M-1)(1-x_0^2)}\right. \\ &\quad \left. -\{a + b(M-2)\}x_0 + (a-b)(M-2)s_2\right]^2. \end{aligned} \quad (14)$$

s_1, s_2, and x_0 can be obtained by maximizing P_S of Eq.(14). After some algebra, we have $s_1 = \mathrm{sgn}(b) := b/|b|$ and $s_2 = -1$. Also, it follows that x_0 is a solution to the following quartic polynomial:

$$\sum_{k=0}^{4} f_k x_0^k = 0, \quad (15)$$

where $f_0, \cdots, f_4$ are expressed by Eq.(16).

Since the quartic polynomials can be solved analytically, one can obtain an analytical solution to Eq. (15). We omit the analytical expression because of its complexity.

IV. Dolinar-like receiver for PPM optical coherent signals

A. Configuration of Dolinar-like receiver

Let us consider a Dolinar-like near-optimal receiver for PPM optical coherent signals expressed by Eq. (3) with equal prior probabilities. The PPM coherent signals $\{|\Psi'_m\rangle\}_{m\in\mathcal{I}_M}$ are expressed by

$$|\Psi'_m\rangle = \bigotimes_{n=0}^{M-1}|\alpha_n\rangle = |\alpha_0\rangle\otimes\cdots\otimes|\alpha_{M-1}\rangle \quad (17)$$

with $\alpha_k = \alpha\delta_{k,m}$, where $\delta_{k,m}$ is the Kronecker delta, $|\alpha_k\rangle$ ($= |\alpha\rangle$ or $|0\rangle$) is a coherent state, and α is a positive real number. We have

$$\begin{aligned} |\Psi'_0\rangle &= |\alpha\rangle\otimes|0\rangle\otimes\cdots\otimes|0\rangle, \\ |\Psi'_1\rangle &= |0\rangle\otimes|\alpha\rangle\otimes\cdots\otimes|0\rangle, \\ &\vdots \\ |\Psi'_{M-1}\rangle &= |0\rangle\otimes|0\rangle\otimes\cdots\otimes|\alpha\rangle. \end{aligned} \quad (18)$$

Since the inner product of the two coherent states $|\alpha\rangle$ and $|0\rangle$ is $\langle\alpha|0\rangle = e^{-\alpha^2/2}$, the inner product $\langle\Psi'_j|\Psi'_k\rangle = K$ with distinct j and k is

$$K = \langle\alpha|0\rangle\langle 0|\alpha\rangle = e^{-\alpha^2}. \quad (19)$$

This receiver, which is based on feedback control, can be expressed by a sequential measurement in which the time duration of the measurement, normalized to $0 < t \leq 1$, is split into N intervals $\{t_{j-1} < t \leq t_j\}_{j\in\{1,\cdots,N\}}$, where $t_j := j/N$. The receiver performs a collective measurement with M measurement operators during each time interval $t_{j-1} < t \leq t_j$, where one adapts subsequent collective measurements based on the results of the previous ones. We can expand the coherent state $|\Psi'_m\rangle$ as $|\Psi'_m\rangle = |\psi'_m\rangle^{\otimes N}$ with

$$|\psi'_m\rangle = \left|\frac{\alpha_0}{\sqrt{N}}\right\rangle\otimes\left|\frac{\alpha_1}{\sqrt{N}}\right\rangle\otimes\cdots\otimes\left|\frac{\alpha_{M-1}}{\sqrt{N}}\right\rangle. \quad (20)$$

According to the Bayesian updating scheme (see Ref. [20]), the posterior probabilities $\xi_m^{(j)}$ of the state $|\Psi'_m\rangle$ after performing the j-th measurement is determined as

$$\xi_m^{(j)} := \begin{cases} P_\mathrm{S}^{(j)}, & m = o(j), \\ \dfrac{1-P_\mathrm{S}^{(j)}}{M-1}, & \text{otherwise}, \end{cases} \quad (21)$$

where $P_\mathrm{S}^{(j)}$ is the average success probability at the time t_j and $o(j) \in \mathcal{I}_M$ is the result of the j-th measurement. As the j-th measurement, we perform a minimum-error measurement for $\{|\psi'_m\rangle\}_{m\in\mathcal{I}_M}$ with prior probabilities $\{\xi_m^{(j-1)}\}_{m\in\mathcal{I}_M}$. Since $\{|\psi'_m\rangle\}_{m\in\mathcal{I}_M}$ are pure states with equal real inner products, a minimum-error measurement for them can be derived as shown in Sec. III. Figure 1 shows a schematic representation of a Dolinar-like receiver for ternary PPM coherent signals $\{|\Psi'_m\rangle\}_{m\in\mathcal{I}_3}$. Note that the proposed receiver can be considered to be a generalization

$$
\begin{aligned}
f_4 &= -q^2M^4 - 2q(2qK^2 - 2K^2 - qK + K - 2q)M^3 \\
&\quad + (11q^2K^2 - 10qK^2 - K^2 - 8q^2K + 8qK - 4q^2 - 2q)M^2 \\
&\quad - 2(4q^2K^2 - 2qK^2 - 2K^2 - 4q^2K + 3qK + K - 2q)M + 4qK^2 - 4K^2 - 4qK + 4K - 1, \\
f_3 &= -2(q-1)(M-2)/M\{\sqrt{1-K}(M-1)\sqrt{KM-K+1} \\
&\quad \times (2qKM^2 - qM^2 - 3qKM - KM + 2qM + 2K - 1) \\
&\quad + (K-1)(2qKM^3 - 6qKM^2 + qM^2 + 5qKM + KM - 2qM - 2K + 1)\}, \\
f_2 &= (M-1)\{q^2M^3 + (5q^2K^2 - 6qK^2 + K^2 - 3q^2K + 4qK - K - 3q^2)M^2 \\
&\quad + (-12q^2K^2 + 16qK^2 - 4K^2 + 12q^2K - 18qK + 6K + 4q - 1)M \\
&\quad + 8q^2K^2 - 12qK^2 + 4K^2 - 12q^2K + 20qK - 8K + 4q^2 - 8q + 3\}, \\
f_1 &= 2(q-1)(M-2)/M^3[\sqrt{1-K}(M-1)\sqrt{KM-K+1}\{(2K-1)qM^4 - (6qK + K - 4q)M^3 \\
&\quad + (9qK + 3K - 8q - 1)M^2 + (-8qK - 3K + 8q + 2)M + 4K - 4\} \\
&\quad + (K-1)\{3qKM^5 + q(-14K + 3)M^4 + (25qK + 3K - 12q)M^3 \\
&\quad + (-21qK - 9K + 16q + 3)M^2 + (8qK + 9K - 8q - 6)M - 4K + 4\}], \\
f_0 &= 1/M^4[-2\sqrt{1-K}(M-2)(M-1)\sqrt{KM-K+1}\{q^2(2K-1)M^5 + q(-9qK - K + 6q)M^4 \\
&\quad + (15q^2K + 4qK + K - 13q^2 - 1)M^3 + (-10q^2K - 7qK - 3K + 10q^2 + 4q + 2)M^2 \\
&\quad + (8qK + 2K - 8q - 1)M - 2K + 2\} + \{q^2(K-1)KM^8 \\
&\quad + (-15q^2K^2 + 2qK^2 - K^2 + 16q^2K - 2qK + K - 2q^2)M^7 \\
&\quad + (72q^2K^2 - 8qK^2 + 7K^2 - 88q^2K + 12qK - 8K + 19q^2 - 2q + 1)M^6 \\
&\quad + (-168q^2K^2 - 20K^2 + 235q^2K - 14qK + 25K - 70q^2 + 8q - 6)M^5 \\
&\quad + (211q^2K^2 + 52qK^2 + 32K^2 - 336q^2K - 46qK - 43K + 126q^2 + 14)M^4 \\
&\quad + (-140q^2K^2 - 108qK^2 - 38K^2 + 252q^2K + 146qK + 53K - 112q^2 - 40q - 18)M^3 \\
&\quad + (40q^2K^2 + 92qK^2 + 37K^2 - 80q^2K - 156qK - 54K + 40q^2 + 64q + 18)M^2 \\
&\quad - 8(K-1)(4qK + 3K - 4q - 2)M + 8(K-1)^2\}]. \qquad (16)
\end{aligned}
$$

of the Dolinar receiver; indeed, they are the same when $M = 2$. We assume that a minimum-error measurement for $\{|\psi'_m\rangle\}_{m\in\mathcal{I}_M}$ can be realized or approximately realized by practical optical devices. A possible example of a concrete realization of a Dolinar-like receiver is shown in Fig. 2. The realization of their minimum-error measurement is left for future work.

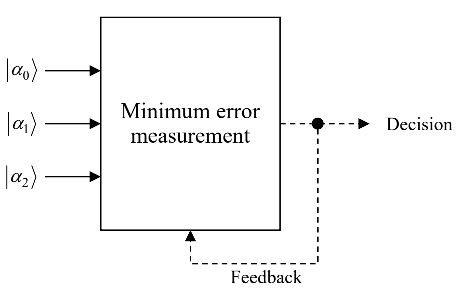

Fig. 1. Dolinar-like receiver with feedback circuit.

B. Another application

The Dolinar-like receiver described in the previous subsection can be applied to quantum states with equal real inner products that are not PPM coherent signals if some preprocessing is allowed. As an example, Fig. 3 shows an optical circuit that transforms the quantum states $\{|\Phi_m\rangle\}_{m\in\mathcal{I}_8}$ encoded by the $(7,3)$ simplex code into PPM coherent signals $\{|\Psi'_m\rangle\}_{m\in\mathcal{I}_8}$. In this example, the input state is expressed by $|\Phi_m\rangle = \bigotimes_{n=0}^{6}|\alpha_n\rangle$, where $|\alpha_n\rangle = |0\rangle$ holds if the corresponding symbol is 0;

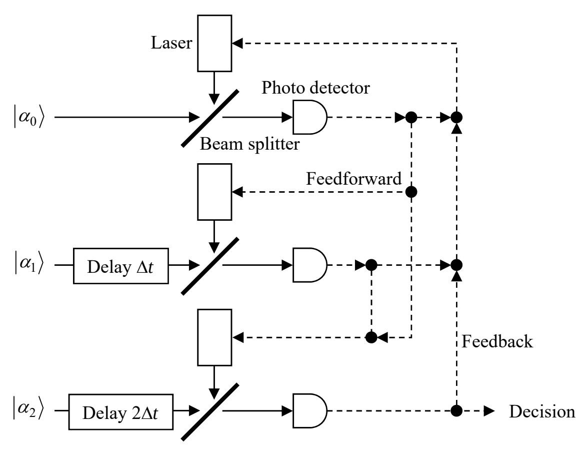

Fig. 2. Possible example of configuration of Dolinar-like receiver

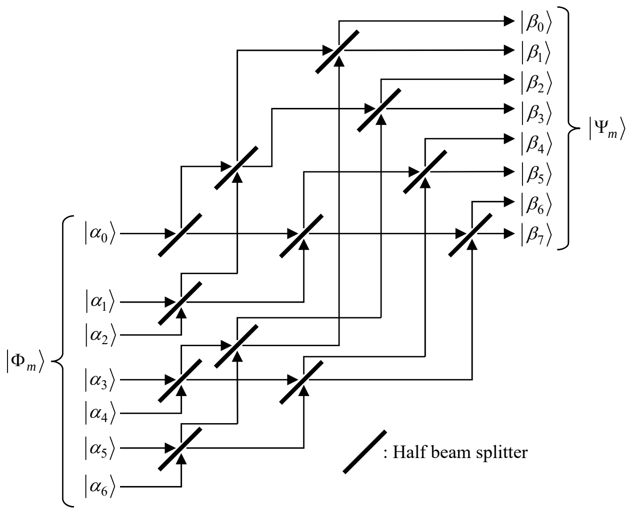

Fig. 3. Converting (7,3) simplex codeword to PPM coherent signal.

otherwise, $|\alpha_n\rangle = |\alpha\rangle$. $|\Psi'_m\rangle$ can be expressed by

$$|\Psi'_m\rangle = |\beta_0\rangle \otimes |\beta_1\rangle \otimes |\beta_2\rangle \otimes |\beta_3\rangle \otimes |-\beta_4\rangle \otimes |-\beta_5\rangle \otimes |-\beta_6\rangle \otimes |\sqrt{2}\alpha - \beta_7\rangle . \quad (22)$$

Assume that each half beam splitter in Fig. 3 has the property such that when the coherent lights $|\alpha_\mathrm{L}\rangle$ and $|\alpha_\mathrm{D}\rangle$ are respectively incident to the left and bottom sides, the coherent lights of $\left|(\alpha_\mathrm{L} + \alpha_\mathrm{D})/\sqrt{2}\right\rangle$ and $\left|(\alpha_\mathrm{L} - \alpha_\mathrm{D})/\sqrt{2}\right\rangle$ are respectively emitted from the right and upper sides. Then, β_k $(k \in \mathcal{I}_8)$ can be expressed as

$$\begin{aligned}
\beta_0 &= (\alpha_0 - \alpha_1 + \alpha_2 - \alpha_3 + \alpha_4 - \alpha_5 + \alpha_6)/\sqrt{8},\\
\beta_1 &= (\alpha_0 - \alpha_1 + \alpha_2 + \alpha_3 - \alpha_4 + \alpha_5 - \alpha_6)/\sqrt{8},\\
\beta_2 &= (\alpha_0 + \alpha_1 - \alpha_2 - \alpha_3 + \alpha_4 + \alpha_5 - \alpha_6)/\sqrt{8},\\
\beta_3 &= (\alpha_0 + \alpha_1 - \alpha_2 + \alpha_3 - \alpha_4 - \alpha_5 + \alpha_6)/\sqrt{8},\\
\beta_4 &= (\alpha_0 - \alpha_1 - \alpha_2 - \alpha_3 - \alpha_4 + \alpha_5 + \alpha_6)/\sqrt{8},\\
\beta_5 &= (\alpha_0 - \alpha_1 - \alpha_2 + \alpha_3 + \alpha_4 - \alpha_5 - \alpha_6)/\sqrt{8},\\
\beta_6 &= (\alpha_0 + \alpha_1 + \alpha_2 - \alpha_3 - \alpha_4 - \alpha_5 - \alpha_6)/\sqrt{8},\\
\beta_7 &= (\alpha_0 + \alpha_1 + \alpha_2 + \alpha_3 + \alpha_4 + \alpha_5 + \alpha_6)/\sqrt{8}. \quad (23)
\end{aligned}$$

Let $c_m := \beta_m/\sqrt{2}\alpha$ $(m \in \{0,1,2,3\})$, $c_m := -\beta_m/\sqrt{2}\alpha$ $(m \in \{4,5,6\})$, and $c_7 := 1 - \beta_7/\sqrt{2}\alpha$. For each $(7,3)$ simplex codeword, c_m with $m \in \mathcal{I}_8$ is expressed by Table I, which implies that $\{|\Psi'_m\rangle\}$ are PPM coherent signals.

V. Evaluation

We numerically evaluated the performance of the Dolinar-like receiver for PPM coherent signals $\{|\Psi'_m\rangle\}_{m\in\mathcal{I}_M}$. Figure 4 shows the average error probability P_E with respect to the average photon number $S := |\alpha|^2$ of $|\Psi'_m\rangle$. We set the number of intervals as $N = 10^5$. At each time interval $t_{j-1} < t \le t_j$, a minimum-error measurement for $\{|\psi'_m\rangle\}_{m\in\mathcal{I}_M}$ was performed and their posterior probabilities $\{\xi_m^{(j)}\}_{m\in\mathcal{I}_M}$ were computed using

TABLE I
Converting (7,3) simplex codeword to PPM coherent signal.

State vector	codeword	$c_0c_1c_2c_3c_4c_5c_6c_7$
$\lvert\Psi'_0\rangle$	0000000	00000001
$\lvert\Psi'_1\rangle$	1010101	10000000
$\lvert\Psi'_2\rangle$	0110011	00000100
$\lvert\Psi'_3\rangle$	1100110	00100000
$\lvert\Psi'_4\rangle$	0001111	00000010
$\lvert\Psi'_5\rangle$	1011010	01000000
$\lvert\Psi'_6\rangle$	0111100	00001000
$\lvert\Psi'_7\rangle$	1101001	00010000

Eq. (21). Note that, in a preliminary experiment, we obtained almost the same performance when $N = 10^3$. The quantum limit (i.e., the average error probability of a minimum-error measurement) and the performance of the receiver proposed by Yamazaki [21] (referred to as the conventional receiver) are also plotted in Fig. 4.

As M and S increase, the gap between the performance of the proposed receiver and the quantum limit widens. A possible reason is that the proposed receiver computes the posterior probabilities $\{\xi_m^{(j)}\}_{m\in\mathcal{I}_M}$ using only the last measurement result. However, one can see that the proposed receiver achieves performance close to the quantum limit when S is small (e.g., $S \le 1$). In a separate experiment, we verified that the average error probability of the proposed receiver was less than 1.04 times of the quantum limit for any M with $M \le 10$ and $S \le 1$.

VI. Conclusion

We derived an analytical expression for a minimum-error measurement for quantum pure states with equal real inner products whose prior probabilities except for one are equal. We also demonstrated in our numerical experiments that a Dolinar-like receiver for PPM optical coherent signals achieves near-optimal performance when the average photon number is small such as less than one.

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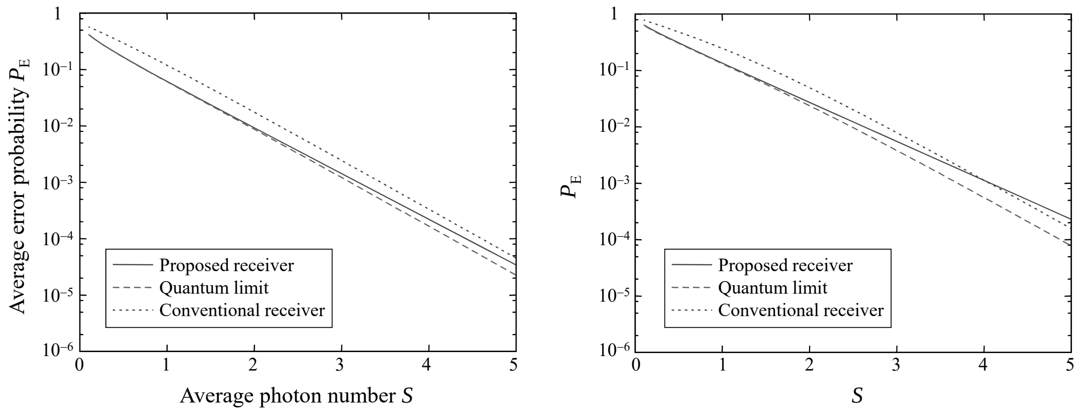
Appendix

A. Supplement of Eq. (1)

For any M pure states $\{|\Psi_m\rangle\}_{m\in\mathcal{I}_M}$, the M-dimensional square matrix whose (j,k) element is the inner product $\langle\Psi_j|\Psi_k\rangle$ is called the Gram matrix of $\{|\Psi_m\rangle\}$. The following lemma holds.

Lemma 1 Let Λ_Ψ and Λ_Φ be the Gram matrices of two sets of M pure states $\{|\Psi_m\rangle\}_{m\in\mathcal{I}_M}$ and $\{|\Phi_m\rangle\}_{m\in\mathcal{I}_M}$, respectively. Then, a necessary and sufficient condition for $\Lambda_\Psi = \Lambda_\Phi$ is that there exists a unitary operator V such that $|\Psi_m\rangle = V|\Phi_m\rangle$ holds for any $m \in \mathcal{I}_M$.



(a) $M = 3$ (b) $M = 8$

Fig. 4. The average error probabilities P_{E} for (a) ternary and (b) 8-ary PPM coherent signals.

Let $\mathcal{H}_\Psi$ and $\mathcal{H}_\Phi$ be the complex Hilbert spaces spanned by $\{|\Psi_m\rangle\}$ and $\{|\Phi_m\rangle\}$, respectively. Let us express $|\Phi_m\rangle = \sum_k |v_k\rangle \langle v_k|\Phi_m\rangle$ using an orthonormal basis $\{|v_m\rangle\}$ satisfying $\mathcal{H}_\Phi \subseteq \mathrm{span}\{|v_m\rangle\}$. It follows from Lemma 1 that if $\Lambda_\Psi = \Lambda_\Phi$ holds, then $|\Psi_m\rangle$ can be written as $|\Psi_m\rangle = \sum_k |u_k\rangle \langle v_k|\Phi_m\rangle$ with an appropriate orthonormal basis $\{|u_m\rangle\}$ satisfying $\mathcal{H}_\Psi \subseteq \mathrm{span}\{|u_m\rangle\}$.

Proof The sufficiency is obvious since $\langle \Psi_j|\Psi_k\rangle = \langle \Phi_j| V^\dagger V |\Phi_k\rangle = \langle \Phi_j|\Phi_k\rangle$ holds. We will prove the necessity.

Assume that $\Lambda_\Psi = \Lambda_\Phi$ holds. We have

$$\dim \mathcal{H}_\Psi = \mathrm{rank}\, \Lambda_\Psi = \mathrm{rank}\, \Lambda_\Phi = \dim \mathcal{H}_\Phi. \quad (24)$$

Let $\Psi := [|\Psi_0\rangle, \cdots, |\Psi_{M-1}\rangle]$, which is $N \times M$ matrix ($N := \dim \mathcal{H}_\Psi$) whose $(m+1)$-th colomn is $|\Psi_m\rangle$. Also, let $\Phi := [|\Phi_0\rangle, \cdots, |\Phi_{M-1}\rangle]$. To prove this lemma, it suffices to show that there exists N-dimensional unitary matrix V satisfying $\Psi = V\Phi$.

Let $V := (\Psi^+)^\dagger \Phi^\dagger$, where Ψ^+ is the Moore-Penrose pseudo-inverse matrix of Ψ; then, we have

$$\begin{aligned} V\Phi &= (\Psi^+)^\dagger \Phi^\dagger \Phi = (\Psi^+)^\dagger \Lambda_\Phi = (\Psi^+)^\dagger \Lambda_\Psi \\ &= (\Psi^+)^\dagger \Psi^\dagger \Psi = \Psi, \end{aligned} \quad (25)$$

which follows from $\Psi\Psi^+ = I$. Also, V satisfies

$$VV^\dagger = (\Psi^\dagger)^+ \Phi^\dagger \Phi \Psi^+ = (\Psi^\dagger)^+ \Psi^\dagger \Psi \Psi^+ = I. \quad (26)$$

From rank $V = N$, there exists V^{-1}. Premultiplying $VV^\dagger = I$ by V^{-1} yields $V^\dagger = V^{-1}$, and thus V is unitary satisfying $\Psi = V\Phi$. ■

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