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***Abstract*—We calculate the Holevo capacity of attenuation channels with phase insensitive amplifiers and find the optimum arrangement for those amplifiers.**

I. Introduction

The ultimate capability of optical communication systems has been revealed by computing the Holevo capacity of bosonic Gaussian channels. An upper bound for the information transmitted by coherent states with thermal noise was anticipated by Gordon [1]. The direct coding theorem for discrete mixed states was given by Holevo [3] and Schumacher-Westmoreland [2] independently, and it was extended to continuous channels with constrained inputs by Holevo [8]. By virtue of these results, it was shown rigorously for the first time that Gordon's upper bound is identical to the operational channel capacity [8]. Basing on Holevo's general treatment of Gaussian states and bosonic channels [5], [6], we obtained the Holevo capacity for noisy attenuation channels with one-mode Gaussian input states [4], in particular for attenuation channels with phase insensitive linear amplifiers (PIAs) or phase sensitive amplifiers (PSAs) [9]. We then imposed the following restrictions:

1) The amplifiers are arranged at equal intervals on an attenuation channel
2) A gain G_i of each amplifier is set so that the amplification can cancel the reduction of signal energy caused by attenuation with a transmittance k_i, i.e. $G_i = 1/k_i$.

In this paper we remove the restriction (1) and consider the optimization problem for arrangement of PIAs.

II. Holevo capacity for attenuation channels with PIAs

We consider an optical communication system where an attenuation channel with transmittance $K = k_1 k_2 \cdots k_n k_{n+1}$ is devided into $n+1$ attenuation channels with a transmittance $k_i (i = 1, \ldots, n+1)$ and PIAs with a gain $G_i = k_i^{-1} (i = 1, \ldots, n)$ are put (see Figure 1).

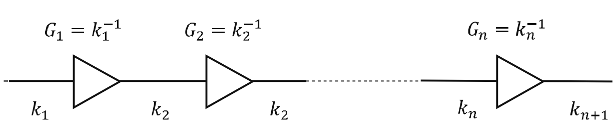

Fig. 1. Schematic diagram of an attenuation channel with PIAs.

We assume the sender uses coherent states as physical carriers conveying classical information. In order to describe changes of coherent states through an attenuation channel with PIAs, it is convenient to introduce an idea of quantum Gaussian state, which is characterized by a mean vector and a correlation matrix [6]. The mean vector and correlation matrix for a coherent state $|\alpha\rangle$, $\alpha = x + iy$, are given as

$$m_\alpha = \sqrt{2\hbar}\begin{pmatrix} x \\ y \end{pmatrix}, \quad A_\alpha = \frac{\hbar}{2} I_2, \tag{1}$$

with the 2×2 identity matrix I_2. The PIA is a usual linear amplifier, which is described by the transformation in the Heisenberg picture,

$$a' = G_i^{1/2} a + (G_i - 1)^{1/2} a_0, \tag{2}$$

where a (resp. a') is an input modal photon annihilation operator (resp. an output one), $a_0^\dagger$ is a creation operator representing the additive noise introduced by the amplification, and $G_i (\geq 1)$ denotes a power gain of amplifier. Let us consider a Gaussian state with a mean m and a correlation matrix A. Then the output state from the PIA with a power gain G_i is the Gaussian state with the mean vector $G_i^{1/2} m$ and the correlation matrix

$$\phi_{G_i}(A) = G_i A + \hbar \frac{G_i - 1}{2} I_2. \tag{3}$$

The PIA preserves the heterodyne signal-to-noise ratio (SNR) at the output [7].

On the other hand an attenuation channel with a transmittance k_i $(0 \leq k_i \leq 1)$ is described by the transformation

$$a' = k_i^{1/2} a + (1 - k_i)^{1/2} a_0 \tag{4}$$

Let us consider a Gaussian state with a mean m and a correlation matrix A again. Then the output state from an attenuation channel with a transmittance k_i has the mean vector $k_i^{1/2} m$ and the correlation matrix

$$\psi_{k_i}(A) = k_i A + \hbar \frac{1 - k_i}{2} I_2. \tag{5}$$

Thus if we input a coherent state $|\alpha\rangle$ for our optical communication system depicted by Fig. 1, we have the Gaussian state with the mean

$$\prod_{i=1}^{n+1} k_i \prod_{i=1}^{n} G_i m_\alpha \tag{6}$$

and the correlation matrix

$$\psi_{k_{n+1}} \circ \phi_{G_n} \circ \psi_{k_n} \circ \cdots \circ \phi_{G_1} \circ \psi_{k_1}(A_\alpha) \tag{7}$$

as the output. Here $\circ$ denotes the composition of functions. In particular, as $G_i = k_i^{-1}$ holds for our system, the mean vector (6) is $k_{n+1}^{1/2} m_\alpha$ and the correlation matrix is

$$\psi_{k_{n+1}} \left(A_\alpha + \hbar \sum_{i=1}^{n} \frac{1}{k_i} - n\hbar \right) = A_\alpha + \hbar N I_2 \tag{8}$$

with

$$N = k_{n+1} \left(\sum_{i=1}^{n} \frac{1}{k_i} - n \right). \tag{9}$$

We evaluate effects of arranging the PIAs by computing the Holevo capacity with an input constraint E. Here we impose the input constraint onto codewords $(\alpha_1, ..., \alpha_M)$ as

$$|\alpha_1|^2 + ... + |\alpha_M|^2 \leq ME. \tag{10}$$

Through the attenuation channel with the PIAs, coherent states $|\alpha\rangle$ are changed into thermal states

$$S_\alpha = \frac{1}{\pi N} \int \exp \left(-\frac{|z - \alpha|^2}{N} \right) |z\rangle\langle z| d^2 z, \tag{11}$$

where N is given by Eq. (9). Thus our setting can be formulated as a classical-quantum channel

$$\alpha \to S_\alpha, \tag{12}$$

with the input constraints (10). For this channel the Holevo capacity is obtained [8] as

$$\begin{aligned} C =& \log \left(1 + \frac{E}{N+1} \right) + (N + E) \log \left(1 + \frac{1}{N+E} \right) \\ & - N \log \left(1 + \frac{1}{N} \right) =: g(E, N). \end{aligned} \tag{13}$$

The value of the Holevo capacity shows the ultimate transmission rate, which is obtained by employing the optimum coding and the optimum entangled measurement. Note that the first term in Eq.(13)

$$C_{het} = \log \left(1 + \frac{E}{N+1} \right) \tag{14}$$

gives the classical capacity for the separative heterodyne measurement.

III. Optimization of arrangement of PIAs

Let us find the values of $k_1, ..., k_n, k_{n+1}$ that maximize the Holevo capacity (13), where we fix the values of the input constraint E and the total transmittance $K(= k_1 \cdots k_n k_{n+1})$. From Eq. (8) we find that the Holevo capacity (13) is equal to that of the attenuation channel which has thermal states with the additive Gaussian noise

$$\hbar \sum_{i=1}^{n} \frac{1}{k_i} - n\hbar, \tag{15}$$

as input states. This means that when k_{n+1} is fixed to a certain value k the values of $k_1, ..., k_n$ minimizing (15) under the constraint $k_1 \cdots k_n = K/k$ maximize the Holevo capacity. Here we have the inequality

$$\sum_{i=1}^{n} \frac{1}{k_i} \geq n \left(\prod_{i=1}^{n} \frac{1}{k_i} \right)^{1/n}, \tag{16}$$

where the equality holds for $k_1 = k_2 = \cdots = k_n$, that is,

$$k_i = \left(\frac{K}{k} \right)^{1/n}, \quad i = 1, \ldots, n. \tag{17}$$

Thus we find that for a fixed $k_{n+1} = k$ the maximum value of the Holevo capacity is given by

$$C_n(k) = g(kE, N_n) \tag{18}$$

with

$$N_n = k \cdot n \left(\left(\frac{k}{K} \right)^{1/n} - 1 \right). \tag{19}$$

Note that $C_n(k)$ converges to

$$C_\infty(k) = g(kE, N_\infty) \tag{20}$$

with

$$N_\infty = k(\log k - \log K), \tag{21}$$

as $n \to \infty$. Similarly for a fixed $k_{n+1} = k$ we obtain the maximum value of the classical capacity for the heterodyne detection as

$$C_{het,n}(k) = \log \left(1 + \frac{E}{f(k) - n} \right), \tag{22}$$

with

$$f(k) = n \left(\frac{k}{K} \right)^{1/n} + \frac{1}{k}. \tag{23}$$

As the function $f(k)$ achieves the maximum value $(n+1)K^{-1/(n+1)}$ when

$$k = K^{1/(n+1)}, \tag{24}$$

it is found that the maximum value of $C_{het,n}(k)$ is given by

$$C_{het,n} = \log \left(1 + \frac{E}{(n+1)K^{-1/(n+1)} - n} \right), \tag{25}$$

when PIAs are arranged at equal intervals on the attenuation channel. Note that as $n \to \infty$ the optimum value of k given by Eq. (24) converges to 0 and $C_{het,n}$ converges to

$$C_{het,\infty} = \log \left(1 + \frac{E}{1 - \log K} \right), \tag{26}$$

which can be compared to the classical capacity of the attenuation channel with no amplifier, $C_{het,0} = \log(1 + KE)$.

Let us see how usage of amplifiers improves the classical capacity and compare its effect with that of quantum collective measurement. Figure 2 shows graphs of the

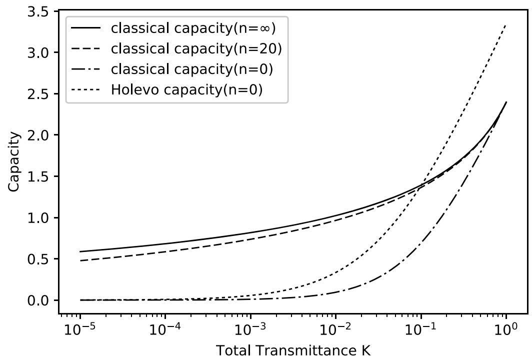

Fig. 2. Dependence of the classical capacity for the heterodyne detection with n=0, 20, ∞ and the Holevo capacity with no amplifier on the total transmittance K when $E = 10$.

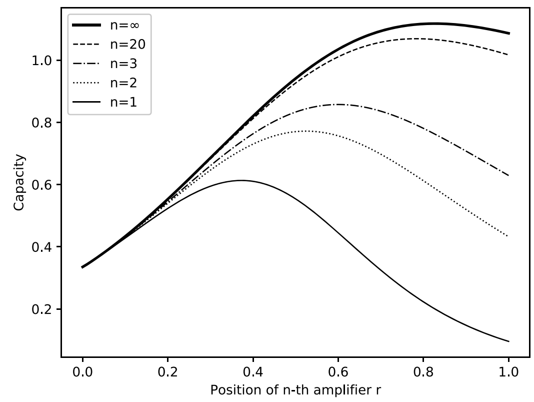

Fig. 3. Dependence of the Holevo capacity of the attenuation channel with n amplifiers on the position r of the last (n-th) amplifier when $K = 0.01$ and $E = 10$.

classical channel capacities $C_{het,n}$ $(n = 0, 20, \infty)$ and the Holevo capacity without amplifiers, $g(KE, 0)$, with respect to the total transmittance K, where an input signal energy E is equal to 10. Here the solid line represents the classical capacity $C_{het,\infty}$, the dashed line the classical capacity $C_{het,20}$, the dashdot line the classical capacity without amplifiers, $C_{het,0}$, the dotted line the Holevo capacity without amplifiers. These graphs indicate that (i) 20 amplifiers are almost enough for the heterodyne detection and (ii) the Holevo capacity without amplifiers exceeds the classical capacity for the heterodyne detection with amplifiers when transmittance $K > 10^{-1}$. Note that this tendency holds for other values of E.

It is difficult to obtain the value of k which maximizes the Holevo capacity $C_n(k)$ analytically and we rely on a numerical computation. Let us consider the ratio r of distance between the sender and n-th amplifier to that between the sender and the receiver. In the following we use the ratio r instead of the transmittance k as a parameter in order to help our intuitive understanding. The relation between k and r is given as $k = K^{1-r}$. Remark that in our setting $r = 0$ means that we do not use any amplifiers. Figure 3 shows graphs of Holevo capacities $C_n(K^{1-r})$ of attenuation channels with n amplifiers ($n = 1, 2, 3, 20, \infty$), when a total transmittance is $K = 0.01$ and an input signal energy is $E = 10$. In the figure the horizontal axis represents the ratio r corresponding to a position of the last (n-th) amplifier. This figure shows that the optimum positions take smaller values than those for the heterodyne detection, which are given by $n/(n+1)$. In particular, when $n = 1$, putting an amplifier near the receiver makes values of the Holevo capacity smaller than that for the attenuation channel with no amplifier, $C_1(K)$. In addition the ratio r giving the optimum position does not converge to 1 as $n \to \infty$ unlike the case of heterodyne detection. As the Holevo capacity is obtained by optimizing the measurement process, putting an amplifier at the receiver

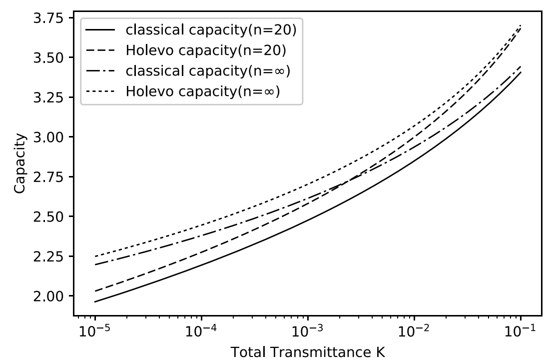

Fig. 4. Dependence of the Holevo capacity and the classical capacity for the heterodyne detection on the total transmittance K when $E = 100$.

($r = 1$) makes it worse. Figure 4 shows graphs of the Holevo capacities C_n and the classical capacities $C_{het,n}$ ($n = 20, \infty$) with respect to the total transmittance K when an input signal energy is $E = 100$. Here the Holevo capacities C_n are given by maximizing $C_n(K^{1-r})$ with respect to r numerically.

IV. Conclusion

We have computed Holevo capacities and classical capacities achieved by arranging PIAs on an attenuation channel optimally. This only gives an elementary analysis on how to enhance the ability of the attenuation channel because the PIA should not be an optimal amplifier. We will study on an effective way of enhancing it in a more general setting in our future work.

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