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***Abstract*—In the literature [23], simple expressions of the random coding and the expurgation exponents for M-ary phase shift keying (PSK) coherent state signal were respectively derived, and their numerical behaviors are illustrated. In this article, a simple expression of the sphere-packing exponent for M-ary PSK coherent state signal is derived, based on the formula of the sphere-packing exponent for classical-quantum channels by Dalai [18], [19]. Using the simple expression, the case of 16-PSK is numerically computed as an example for a better understanding of how the exact value of the reliability function at high rates is determined.**

I. Introduction

The channel coding theorem for classical-quantum channels is one of the central theorems in quantum information theory [1], [2]. Remarkable results in the early days of the study on this theorem can be found in the literatures [3], [4], [5], [6]. In 1996, a significant breakthrough on the channel coding theorem for classical-quantum channels was brought by Hasuladen, Jozsa, Schumacher, Westmoreland, and Wootters [7]. They proved the channel coding theorem for discrete classical-quantum channels with pure states. Shortly thereafter, the channel coding theorem for discrete classical-quantum channels with general states was proved by Holevo [8] and by Schumacher and Westmorelandand [9], independently. Further, this result was extended to channels with constrained inputs by Holevo [10] and the capacity of quantum Gaussian states was calculated by Holevo, Sohma, and Hirota [11].

The proof mentioned above is based on the asymptotic property of typical sequences. Like in the case of classical information theory [12], [13], [14], an alternative approach based on the *reliability function* $E(R)$, which tells us how quickly the decoding error P_{e} vanishes at rates R below the channel capacity C in the codeword length N with the form $P_{\mathrm{e}} \approx \mathrm{e}^{-NE(R)}$, has begun to be investigated in quantum scenario. The random coding exponent $E_r(R)$ (which is a lower bound of the reliability function), the expurgation exponent $E_{ex}(R)$ (a lower bound), and the zero-rate reliability function $E(+0)$ for classical-quantum channel with pure states were formulated in the literature [15] by Burnashev and Holevo. This successfully provides an alternative proof of the channel coding theorem for classical-quantum channel with pure states. The general version of the random coding exponent $E_r(R)$ was conjectured in the literature [15], but still open. The general version of the expurgation exponent $E_{ex}(R)$ can be found in the literature [16] by Holevo. The error exponents, $E_r(R)$, $E_{ex}(R)$, and $E(+0)$, for constrained inputs, in particular, for the quantum Gaussian channel in one mode, were precisely investigated in the literature [17] by Holevo, Sohma, and Hirota. As for upper bounds of the reliability function, significant progress was made by Dalai. He gave a formulation of the sphere-packing exponent $E_{sp}(R)$ for discrete classical-quantum channels with general states in line with Shannon-Gallager-Berlekamp's approach [18], [19], and proposed a new framework for lower bound of the decoding error at low rates via the zero-error capacity [19]. Further discussions on the sphere-packing exponent and the associated decoding error bound can be found in the literatures [20], [21], [22].

By the parallel use of Dalai's formula of the sphere-packing exponent $E_{sp}(R)$ and Burnashev-Holevo's formula of the random coding exponent $E_r(R)$, the reliability function $E(R)$ for pure state channels at high rates was exactly determined. In the literature [23] by the author, simple expressions of the random coding and the expurgation exponents for M-ary phase shift keying (PSK) coherent state signal were reported. As an extension of this preceding work, the derivation of a simple expression of the sphere-packing exponent $E_{sp}(R)$ for M-ary PSK is of natural interest as well as computation of the exact value of the reliability function for M-ary PSK by the parallel use of the random coding exponent and it. In line with this interest, a simple expression of the sphere-packing exponent $E_{sp}(R)$ for M-ary PSK will be shown and a numerical calculation of the reliability function $E(R)$ will be done in the case of 16-PSK coherent state signal.

II. A Discrete Classical-Quantum Channel with Pure States

Consider a discrete classical-quantum channel with pure states, $k \in \mathcal{A} \mapsto |\psi_k\rangle \in \mathcal{B}$, having an input alphabet $\mathcal{A} = \{1, 2, \ldots, M\}$ and an output alphabet $\mathcal{B} = \{|\psi_1\rangle, |\psi_2\rangle, \cdots, |\psi_M\rangle\}$ that consists of M state vectors in a Hilbert space $\mathcal{H}$ under consideration. A prior probability distribution $\mathbf{p} = (p_1, p_2, \ldots, p_M)$ on the input alphabet $\mathcal{A}$ corresponds to coding for this channel. According to the channel coding theorem for pure state channels [7], [15], the channel capacity for this discrete classical-quantum channel with pure states is given by

$$C = \max_{\mathbf{p}} \left[H(\sum_{k=1}^{M} p_k |\psi_k\rangle\langle\psi_k|) \right], \quad (1)$$

where $H(\hat{\rho}) \equiv -\mathrm{Tr}\hat{\rho}\ln\hat{\rho}$ is the von Neumann entropy for a density operator $\hat{\rho}$.

The reliability function is defined by

$$E(R) \equiv \limsup_{N\to\infty} \left[\frac{-\ln P_{\mathrm{e}}(N,R)}{N} \right] \quad (2)$$

for $0 < R < C$, where N is the length of a codeword, R the transmission rate, and $P_{\mathrm{e}}(N,R)$ the minimum probability of decoding error. For a discrete classical-quantum channel with pure states, the minimum probability of decoding error can be written as

$$P_{\mathrm{e}}(N,R) = \inf_{\mathbb{W},\mathbb{X}} \left[\frac{1}{M'} \sum_{i=1}^{M'} (1 - \langle \tilde{\mathbf{W}}_i | \hat{\mathbf{X}}_i | \tilde{\mathbf{W}}_i \rangle) \right], \quad (3)$$

where $\mathbb{W}$ is a codebook defined by

$$\mathbb{W} = \Big\{ |\tilde{\mathbf{W}}_i\rangle = |\psi_i^{(1)}\rangle \otimes |\psi_i^{(2)}\rangle \otimes \cdots \otimes |\psi_i^{(N)}\rangle : i = 1,2,\ldots,M',\ |\psi_i^{(n)}\rangle \in \mathcal{B} \Big\}, \quad (4)$$

and M' is the size of the codebook, and where the decoding process $\mathbb{X}$ is represented by a positive operator-valued measure (POVM)

$$\mathbb{X} = \Big\{ \hat{\mathbf{X}}_j : \ \hat{\mathbf{X}}_j \geq 0 \ \forall j, \ \sum_{\text{all } j} \hat{\mathbf{X}}_j = \hat{1}^{(N)} \Big\}, \quad (5)$$

and $\hat{1}^{(N)}$ is the identity operator on the N-th tensor of the signal Hilbert space $\mathcal{H}^{\otimes N}$.

III. Error Exponents for M-ary PSK Coherent State Signal

From this point, let us focus on the case of M-ary PSK coherent state signal, which is characterized by the output alphabet

$$\mathcal{B} = \Big\{ |\psi_k\rangle = |\alpha \exp[\mathbf{i}\frac{2\pi(k-1)}{M}]\rangle : k = 1,2,\ldots,M \Big\}, \quad (6)$$

where $|\alpha\rangle$ is a coherent state of light having complex amplitude α and $\mathbf{i} = \sqrt{-1}$. The average number of photons per signal for M-ary PSK is given by $|\alpha|^2$, which is independent from the number M.

A. The Random Coding Exponent

According to the literature [23], a simple expression of the random coding exponent for M-ary PSK is given by

$$E_r(R) = \left(-\ln \sum_{k=1}^{M} \lambda_k^2 \right) - R \quad (7)$$

for $0 < R < R_{cr}$, and

$$E_r(R) = \max_{0\leq s\leq 1} \left[\left(-\ln \sum_{k=1}^{M} \lambda_k^{1+s} \right) - sR \right] \quad (8)$$

for $R_{cr} \leq R < C$, where the critical rate R_{cr} of the random coding exponent and the channel capacity C are respectively given by

$$R_{cr} = -\frac{\sum_{k=1}^{M} \lambda_k^2 \ln \lambda_k}{\sum_{k=1}^{M} \lambda_k^2} \quad (9)$$

and

$$C = -\sum_{k=1}^{M} \lambda_k \ln \lambda_k, \quad (10)$$

and where the eigenvalues of the density operator $\hat{\rho}(\mathbf{u}) = (1/M)\sum_{k=1}^{M} |\psi_k\rangle\langle\psi_k|$ for the uniform distribution $\mathbf{u} = (1/M,\ldots,1/M)$ on $\mathcal{A}$ are given by

$$\lambda_k = \frac{1}{M} \sum_{\ell=1}^{M} A_{(1,\ell)} \cos\Big[\Theta_{(1,\ell)} - \frac{2\pi}{M} k(\ell-1) \Big] \quad (11)$$

with

$$A_{(k,\ell)} = \exp\left[-2|\alpha|^2 \sin^2\Big[\frac{\pi}{M}(\ell-k) \Big] \right], \quad (12)$$

$$\Theta_{(k,\ell)} = |\alpha|^2 \sin\Big[\frac{2\pi}{M}(\ell-k) \Big]. \quad (13)$$

B. The Expurgation Exponent

From the literature [23], a simple expression of the expurgation exponent for M-ary PSK is given by

$$E_{ex}(R) = \max_{s\geq 1} \left[-sR - s\ln \frac{1}{M} \sum_{\ell=1}^{M} \exp\left[-\frac{4|\alpha|^2}{s} \tilde{z}_{(1,\ell)} \right] \right] \quad (14)$$

for $0 < R \leq R'_{cr}$, and

$$E_{ex}(R) = -\ln \frac{1}{M} \sum_{\ell=1}^{M} \exp\left[-4|\alpha|^2 \tilde{z}_{(1,\ell)} \right] - R \quad (15)$$

for $R'_{cr} < R < R_c$, and $E_{ex}(R) = 0$ for $R_c \leq R < C$, where the critical rate R'_{cr} of the expurgation exponent and the cutoff rate R_c [24] are respectively given by

$$R'_{cr} = -\ln \sum_{k=1}^{M} \lambda_k^2 - \frac{4|\alpha|^2}{M} \cdot \frac{\sum_{\ell=1}^{M} \tilde{z}_{(1,\ell)} \exp[-4|\alpha|^2 \tilde{z}_{(1,\ell)}]}{\sum_{k=1}^{M} \lambda_k^2} \quad (16)$$

and

$$R_c = -\ln \frac{1}{M} \sum_{\ell=1}^{M} \exp[-4|\alpha|^2 \tilde{z}_{(1,\ell)}] \tag{17}$$

with $\tilde{z}_{(k,\ell)} = \sin^2[\pi(\ell - k)/M]$.

C. The Sphere-Packing Exponent

The general formulation of the sphere-packing exponent for classical-quantum channels is given in the literatures [18], [19]. For a pure state alphabet $\mathcal{B} = \{|\psi_k\rangle : 1 \leq k \leq M\}$, the sphere-packing exponent is reduced into the form

$$E_{sp}(R) = \sup_{s \geq 0}[E_0(s) - sR] \tag{18}$$

for $R_\infty < R < C$, where

$$E_0(s) = \max_{\mathbf{p}} E_0(s, \mathbf{p}), \tag{19}$$

$$E_0(s, \mathbf{p}) = -\ln \mathrm{Tr} \left(\sum_k p_k |\psi_k\rangle\langle\psi_k| \right)^{1+s} \tag{20}$$

and where

$$R_\infty = -\ln \min_{\mathbf{p}} \lambda_{\max}(\sum_k p_k |\psi_k\rangle\langle\psi_k|) \tag{21}$$

and $\lambda_{\max}(\cdot)$ denotes the largest eigenvalue of the argument. Our task is to derive a simple expression of the sphere-packing exponent for M-ary PSK.

As shown in Eq.(19), the formula of the sphere-packing exponent involves the maximization problem with respect to the input distribution $\mathbf{p}$. Fortunately, the problem is the same as in the case of random coding exponent. With the same manner used in the literature [23], one can find that the maximizer of this problem for M-ary PSK is the uniform distribution $\mathbf{u} = (1/M, \ldots, 1/M)$. Therefore, the sphere-packing exponent for M-ary PSK coherent state signal is expressed as

$$E_{sp}(R) = \max_{s \geq 0} \left[\left(-\ln \sum_{k=1}^{M} \lambda_k^{1+s} \right) - sR \right] \tag{22}$$

for $R_\infty < R < C$, where the rate R_∞ is formulated as in Eq.(21) and the channel capacity C has been given by Eq.(10). To find the closed-form expression of R_∞ for M-ary PSK, we use the convexity of the largest eigenvalue of density operators of the output states. The convexity of the largest eigenvalue is a well-known property (*e.g.* [25]). To verify this property in our situation, let us define $F(\hat{\rho}) \equiv \lambda_{\max}(\hat{\rho})$, where $\lambda_{\max}(\hat{\rho})$ denotes the largest eigenvalue of a density operator $\hat{\rho}$. For density operators $\hat{\rho}_1$ and $\hat{\rho}_2$, we observe $F(\hat{\rho}_1)\hat{1} - \hat{\rho}_1 \geq 0$ and $F(\hat{\rho}_2)\hat{1} - \hat{\rho}_2 \geq 0$, respectively. Therefore

$$\Big\{ tF(\hat{\rho}_1) + (1-t)F(\hat{\rho}_2) \Big\} \hat{1} - \Big\{ t\hat{\rho}_1 + (1-t)\hat{\rho}_2 \Big\} \geq 0$$

holds for $0 \leq t \leq 1$. Taking a quadratic form of the left-hand side by a normalized eigenvector $|\varphi\rangle$ belonging to the largest eigenvalue of $t\hat{\rho}_1 + (1-t)\hat{\rho}_2$, we have

$$tF(\hat{\rho}_1) + (1-t)F(\hat{\rho}_2) \geq F(t\hat{\rho}_1 + (1-t)\hat{\rho}_2).$$

Thus the function F is a convex function on the set of density operators.

Next, we use this convexity to find the minimizer of Eq.(21) in the case of M-ary PSK. For an arbitrary chosen distribution $\mathbf{p} = (p_1, \ldots, p_M)$, we have

$$\begin{aligned} F(\hat{\rho}(\mathbf{p})) &= \sum_{m=1}^{M} \frac{1}{M} F(\hat{\rho}(\mathbf{p})) \\ &= \sum_{m=1}^{M} \frac{1}{M} F(\hat{\rho}(\mathbf{p}^{(m)})) \\ &\geq F(\sum_{m=1}^{M} \frac{1}{M} \hat{\rho}(\mathbf{p}^{(m)})) \\ &= F(\sum_{k=1}^{M} \frac{1}{M} |\psi_k\rangle\langle\psi_k|) \\ &= F(\hat{\rho}(\mathbf{u})), \end{aligned} \tag{23}$$

where $\hat{\rho}(\mathbf{p})$ and $\mathbf{p}^{(m)}$ are defined as in the literature [23]. Thus the uniform distribution $\mathbf{u}$ minimizes the largest eigenvalue. Therefore the rate R_∞ for M-ary PSK is given by

$$R_\infty = -\ln \max\{\lambda_k : 1 \leq k \leq M\}, \tag{24}$$

where λ_k have been given by Eq.(11). Since $-\ln \lambda_{\max} \leq -\ln \lambda_k$, we obtain $R_\infty \leq R_{cr}$.

Fig. 1 (a) shows the rate R_∞ of the case of 16-PSK ($M = 16$) for $0 < |\alpha|^2 \leq 10$, and Fig. 1 (b) the associated eigenvalues λ_k, $1 \leq k \leq 16$, of the density operator $\hat{\rho}(\mathbf{u})$. In Fig. 1 (b), the envelope curve (black:online) indicates the largest eigenvalue of $\hat{\rho}(\mathbf{u})$. The shape of this envelope determines the shape of R_∞ in Fig. 1 (a). To justify the claim that the minimizer of Eq.(21) is the uniform distribution $\mathbf{u}$, a simple numerical simulation was done. In the simulation, the largest eigenvalues of density operators with randomly generated input distribution were calculated. The simulation result is shown in Fig. 1 (c). In this simulation, 30 random samples were generated for each $|\alpha|^2$, varying from $|\alpha|^2 = 0.05$ to $|\alpha|^2 = 10.00$ with step size 0.05. The solid curve is the largest eigenvalue of $\hat{\rho}(\mathbf{u})$, which is identical to the envelope curve in Fig. 1 (b), and the dots (orange:online) are the samples. We numerically observe that the largest eigenvalue of $\hat{\rho}(\mathbf{u})$ is always less than any largest eigenvalue of the density operator having a non-uniform distribution.

IV. The Reliability Function of M-ary PSK Coherent State Signal at a High Rate Region

Like in the case of classical channel, one can find the exact value of the reliability function of M-ary PSK

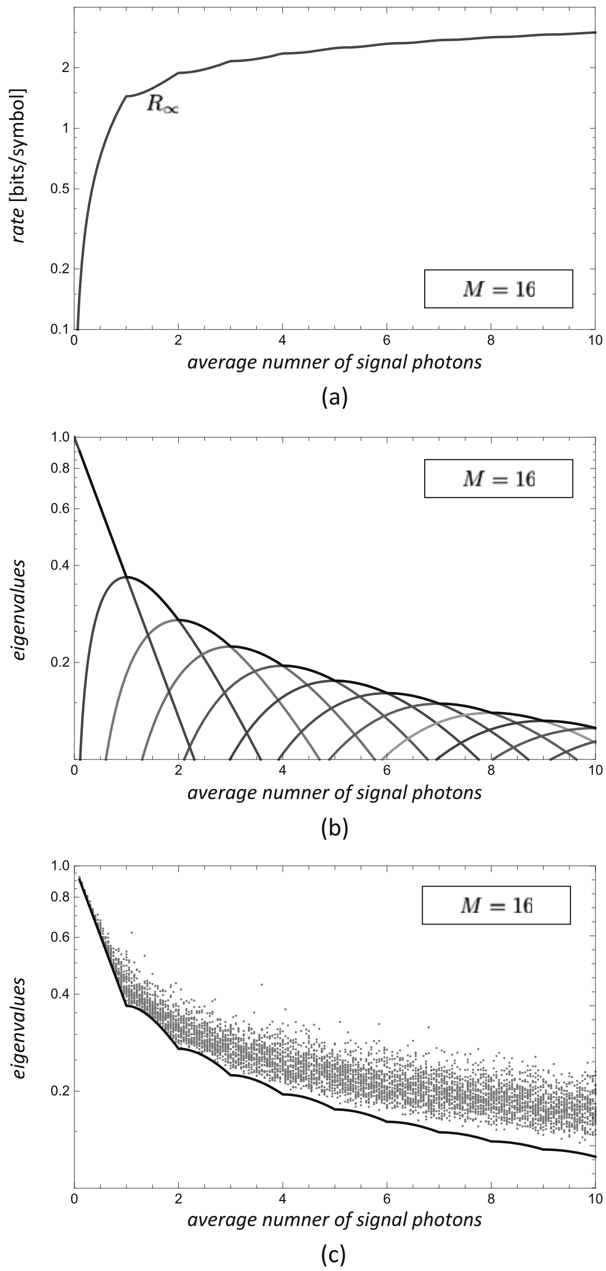

Fig. 1. (a) R_∞. (b) λ_k, $1 \leq k \leq 16$, and $\lambda_{\max}(\hat{\rho}(\mathbf{u}))$. (c) $\lambda_{\max}(\hat{\rho}(\mathbf{u}))$ and $\lambda_{\max}(\hat{\rho}(\mathbf{p}))$, $\mathbf{p} \neq \mathbf{u}$.

coherent state signal at a high-rate region, $R_{cr} < R < C$, by the sandwich of the random coding and the sphere-packing exponents. To illustrate this, the case of $M = 16$ and $|\alpha|^2 = 0.5$ is shown in Fig. 2. The figure (a) shows the random coding, the expurgation, and the sphere-packing exponents. Optimal values of parameter s for the sphere-packing exponent is plotted in Fig. 2 (b), together with the optimal s for the random coding exponent. Fig. 2 (c) shows the optimal s for the expurgation exponent. Some critical values at $M = 16$ and $|\alpha|^2 = 0.5$ are shown in Table I. As mentioned in Remark 5 of the

TABLE I
SOME CRITICAL VALUES AT $M = 16$ AND $|\alpha|^2 = 0.5$

Rate [bits/symbol]		Exponent in $e^{-nE(R)}$	
C	1.33830	~ 0	
R_c	1.10234	> 0	
R_{cr}	0.95780	$E(R_{cr})$	0.10029
R_∞	0.72135	$E_{sp}(R_\infty)$	0.50000
R'_{cr}	0.30365	$E_r(R'_{cr})$	0.55369
$R \to +0$		$E(+0)$	1.00000

literature [19], the bound $E_{sp}(R_\infty) \leq R_\infty$ is tight in general, in the sense that $E_{sp}(R_\infty) = R_\infty$ is possible. In Table I, we observe $R_\infty = 0.72135 \times \ln(2) = 0.50000$ [nats/symbol] $\approx E_{sp}(R_\infty)$, numerically. The same observation about the tightness was confirmed in the range $0 < |\alpha|^2 < 10.00$ through a numerical calculation. The curve in the high rate region $R_{cr} < R < C$ in Fig. 2 (a) is the exact value of the reliability function $E(R)$ of M-ary PSK coherent state signal because the upper and the lower bounds of the reliability function are identical in this region. Thus the exact value of the reliability function $E(R)$ for M-ary PSK can be computed from Eqs.(9), (10), and (22). Furthermore, we observe in Fig.2 (a) that the difference between $E_{sp}(R)$ and $E_r(R)$ is small in the region of $R \approx R_{cr}$ when $R > R_{cr}$. Therefore, the simplified expression (22) may be used to approximate the exact value of $E(R)$ in the region $R \gtrapprox R_{cr}$.

V. CONCLUSION

In this article, a simple expression of the sphere-packing exponent $E_{sp}(R)$ for M-ary PSK coherent state signal was derived as in Eq.(22), together with the associated critical rate R_∞ of Eq.(24). Using this simple expression together with Eqs.(9) and (10), the case of 16-PSK coherent state signal was numerically computed to see the exact value of the reliability function $E(R)$ at the high rate region from the critical rate R_{cr} of the random coding exponent to the chapacity C.

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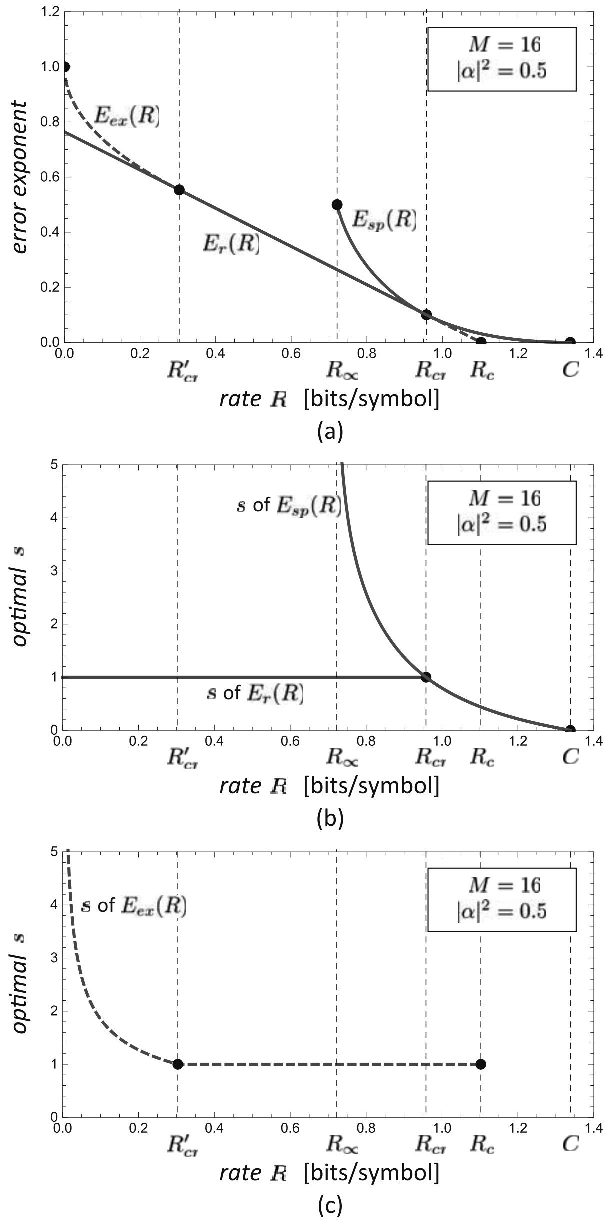

Fig. 2. (a) The random coding exponent, the expurgation exponent, and the zero-rate reliability function for 16-PSK coherent state signal; (b) The optimal s for the random coding exponent; (c) The optimal s for the expurgation exponent.

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