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An Expression for SU(2) Rotations

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***Abstract*—Segercrantz's decomposition of an arbitrary rotation into three consecutive rotations (1966) generalizes the well-known decomposition of the rotation in terms of Euler angles. In that result, rotations have been expressed in terms of an algebra which does not seem to be widely known. In this work, it is pointed out that this expression is closely related to an expression for rotations as SU(2) elements in order to increase accessibility to Segercrantz's work (1966).**

I. Introduction

Mathematics on rotations is fundamental in many fields including mechanics, quantum physics, control theory, aerodynamics, celestial mechanics and so on. Motivated by some issues on quantum computation, the present author has obtained a concrete expression for the minimum number of rotations required for constructing an arbitrarily given target rotation (under some constraint), and more importantly, an algorithm for giving an optimal, i.e., minimum-achieving construction [1]. In the course of obtaining this result, the author carefully checked several basic results on rotations, and explanations of such basics were included in this bulletin when he presented some results on rotations [2], [3], [4], [5]. These results are related to that constructive result [1]; some of them may be viewed as a basis for [1], some may be viewed as remarks to [1], and some may be viewed as by-products of [1].

One of the basics described in these articles [1], [3], [4], [5] is a well-known decomposition of a rotation into three consecutive rotations with Euler angles, or a decomposition due to Segercrantz [6], which generalizes Euler's with flexibility of choices of axes. While this result itself is sometimes mentioned in the literature, it does not seem credited to Segercrantz [6] properly, at least, judging from citations.

Mathematically speaking, a rotation is an element of SO(3). [This notion SO(3), together with a closely related one SU(2), will be described in what follows for those unfamiliar with these notions.] In quantum physics, the use of SU(2) is standard because it suits the theoretical framework of quantum physics well. In some other fields, the use of quaternions, in place of SU(2) or closely related matrices, in treating rotations may be usual.

Returning to credit to Segercrantz [6], that work [6] seems to be often ignored in the literature. This may be because of that article's lack of description of the adopted notation, which has been defined in a publication of an observatory [7], or limited circulation of [7]. Therefore, it would be helpful if the essence of the result of Segercrantz [6] could be understood without resort to 'spinor ring algebra' [7] (and in terms of SU(2) and the Pauli matrices, rather than quaternions, for those who prefers the Pauli matrices to quaternions).

The aim of this memorandum is to give a mathematical expression for rotations, which would be helpful for reading Segercrantz's article [6]. As already mentioned, that article [6] uses quaternions and an unfamiliar notation [7]. On the other hand, this memorandum gives a similar expression for rotations as SU(2) elements, which seems essentially the same as that of Segercrantz's article [6] but is described elementarily without direct resort to that unfamiliar algebra.

This article is organized as follows. In Section II, we fix notation. In Section III, well-known expressions for rotations are reviewed. In Section IV, the expression mentioned above is presented to accomplish the aim of this article. Section VI contains a summary. An appendix is given to describe a well-known relation between SU(2) and SO(3).

II. Definitions

The set of 2×2 unitary matrices with determinant 1 and the set of 3×3 real orthogonal matrices with determinant 1 are denoted by $\mathrm{SU}(2)$ and $\mathrm{SO}(3)$, respectively. They stand for the special unitary group and the special orthogonal group, respectively.[1]

We put $S^2=\{\hat{v}\in\mathbb{R}^3 \mid \|\hat{v}\|=1\}$, and let $\hat{R}_{\hat{v}}(\theta)$ denote the rotation by angle θ about the straight line (through the origin) directed with $\hat{v}$, where the direction of the angle is determined by the rule of right-hand screws.

We let X, Y and Z denote the Pauli matrices:

$$X=\begin{pmatrix}0 & 1\\ 1 & 0\end{pmatrix},\quad Y=\begin{pmatrix}0 & -\mathrm{i}\\ \mathrm{i} & 0\end{pmatrix},\quad Z=\begin{pmatrix}1 & 0\\ 0 & -1\end{pmatrix}.$$

Throughout, I denotes the 2×2 identity matrix. The transpose of a vector $\hat{v}$ is denoted by $\hat{v}^{\mathrm{T}}$.

[1] As can be checked directly and elementarily, $\mathrm{SO}(3)$ stands for the set of rotations in the three-dimensional Euclidean space [8].

III. Well-Known Expressions for the Elements in SU(2)

A. Expression With (a, b)

It can be shown easily that any matrix in $\mathrm{SU}(2)$ can be written as [9]

$$\begin{pmatrix} a & b \\ -b^* & a^* \end{pmatrix} \tag{1}$$

with some complex numbers a and b such that $|a|^2+|b|^2 = 1$.

B. $R_{\hat v}(\theta)$

By the above expression, any matrix in $\mathrm{SU}(2)$ can be written as

$$\begin{pmatrix} w+\mathrm{i}z & y+\mathrm{i}x \\ -y+\mathrm{i}x & w-\mathrm{i}z \end{pmatrix} = wI + \mathrm{i}(xX+yY+zZ) \tag{2}$$

with some real numbers x, y, z and w such that $w^2+x^2+y^2+z^2 = 1$. Take a real number θ such that $\cos(\theta/2) = w$ and $\sin(\theta/2) = \sqrt{1-w^2} = \sqrt{x^2+y^2+z^2}$; write x, y and z as $x = -v_x \sin(\theta/2), y = -v_y \sin(\theta/2)$ and $z = -v_z \sin(\theta/2)$, where $v_x, v_y, v_z \in \mathbb{R}$ and $v_x^2+v_y^2+v_z^2 = 1$. Thus, using real numbers $\theta, v_x, v_y, v_z \in \mathbb{R}$ with $v_x^2+v_y^2+v_z^2 = 1$, any matrix in $\mathrm{SU}(2)$ can be written as

$$(\cos\tfrac{\theta}{2})I - \mathrm{i}(\sin\tfrac{\theta}{2})(v_xX+v_yY+v_zZ). \tag{3}$$

Definition 1:

$$R_{\hat v}(\theta) = (\cos\tfrac{\theta}{2})I - \mathrm{i}(\sin\tfrac{\theta}{2})(v_xX+v_yY+v_zZ) \tag{4}$$

where $\hat v = (v_x, v_y, v_z)^{\mathrm{T}} \in \mathbb{R}^3$ with $\|\hat v\| = \sqrt{v_x^2+v_y^2+v_z^2} = 1$ and $\theta \in \mathbb{R}$, with $\mathbb{R}$ denoting the set of real numbers.

The matrices $R_y(\theta)$ and $R_z(\theta)$ denote the following special cases of $R_{\hat v}$, respectively: $R_y(\theta) = R_{\hat y}(\theta)$, where $\hat y = (0,1,0)^{\mathrm{T}}$, and $R_z(\theta) = R_{\hat z}(\theta)$, where $\hat z = (0,0,1)^{\mathrm{T}}$.

The expression $R_{\hat v}(\theta)$ is useful to see the direct relation between SU(2) and SO(3) (Appendix).

C. Expression With Euler Angles

A better-known parameterization for $\mathrm{SU}(2)$ would be

$$\begin{pmatrix} \mathrm{e}^{-\mathrm{i}\frac{\gamma+\alpha}{2}}\cos\frac{\beta}{2} & -\,\mathrm{e}^{\mathrm{i}\frac{\gamma-\alpha}{2}}\sin\frac{\beta}{2} \\ \mathrm{e}^{-\mathrm{i}\frac{\gamma-\alpha}{2}}\sin\frac{\beta}{2} & \mathrm{e}^{\mathrm{i}\frac{\gamma+\alpha}{2}}\cos\frac{\beta}{2} \end{pmatrix} = R_z(\alpha)R_y(\beta)R_z(\gamma). \tag{5}$$

Here, α, β and γ are real numbers, which are called Euler angles. This parameterization can be obtained by rewriting (1).

IV. Another Expression for SU(2)

For

$$u = \begin{pmatrix} u_1 \\ u_2 \end{pmatrix} \in \mathbb{C}^2,$$

write

$$\tilde u = \begin{pmatrix} -u_2^* \\ u_1^* \end{pmatrix}, \tag{6}$$

$$u^\dagger = (u_1^*, u_2^*) \quad \text{and} \quad \tilde u^\dagger = (-u_2, u_1). \tag{7}$$

Then, we have another expression for SU(2) elements in the following proposition, which may be interpreted as a translation of a known expression used in [6], where the language of ‘spinor ring algebra’ is spoken.

Proposition 1: Any element in $\mathrm{SU}(2)$ can be written as

$$\mathrm{e}^{\mathrm{i}\varphi}uu^\dagger + \mathrm{e}^{-\mathrm{i}\varphi}\tilde u\,\tilde u^\dagger$$

with

$$u = \begin{pmatrix} u_1 \\ u_2 \end{pmatrix} \in \mathbb{C}^2$$

such that $u_1u_1^* + u_2u_2^* = 1$.

The trivial proof may be omitted since we know that by means of spectral decompositions[2] or diagonalizations, any unitary matrix with determinant 1 can be written as

$$\mathrm{e}^{\mathrm{i}\varphi}uu^\dagger + \mathrm{e}^{-\mathrm{i}\varphi}ww^\dagger$$

with some pair of orthogonal unit vectors u and w [10], [11]. Proposition 1 follows from this fact by choosing a particular vector $\tilde u$ as w, which should satisfy $u^\dagger w = 0$.

The relation between this expression and $R_{\hat v}(\theta)$ in Definition 1 can be seen as follows. Given

$$u = \begin{pmatrix} u_1 \\ u_2 \end{pmatrix} \in \mathbb{C}^2$$

with $u_1u_1^* + u_2u_2^* = 1$, we have

$$\begin{aligned} &\mathrm{e}^{\mathrm{i}\varphi}uu^\dagger + \mathrm{e}^{-\mathrm{i}\varphi}\tilde u\,\tilde u^\dagger \\ &\quad= (\cos\varphi)(uu^\dagger + \tilde u\tilde u^\dagger) + \mathrm{i}(\sin\varphi)(uu^\dagger - \tilde u\tilde u^\dagger) \\ &\quad= (\cos\varphi)I \\ &\qquad - \mathrm{i}(\sin\varphi)\begin{pmatrix} v_z & v_x - \mathrm{i}v_y \\ v_x + \mathrm{i}v_y & -v_z \end{pmatrix}, \end{aligned} \tag{8}$$

where

$$v_x = -\mathrm{Re}\,2u_1^*u_2, \quad v_y = -\mathrm{Im}\,2u_1^*u_2$$

and

$$v_z = -(u_1u_1^* - u_2u_2^*).$$

Thus, we have obtained again $R_{\hat v}(\theta)$ in Definition 1 with $\theta = 2\varphi$ from Proposition 1.

V. Segercrantz’s Result

In this section, Segercrantz’s result [6] on rotations is explained. The following theorem is from [6].

Theorem 1: [6, Theorem 1]. Let $\hat n, \hat w, \hat m \in S^2$ be given. Any element in SO(3) can be written as $\hat R_{\hat n}(\alpha)\hat R_{\hat w}(\theta)\hat R_{\hat m}(\gamma)$ for some α, θ, γ if and only if (iff) $\hat w$ is perpendicular to both $\hat n$ and $\hat m$.

While Davenport’s work [12] is often cited as a source of this theorem, the fact is that this result was mentioned in [12] with a comment saying that a reviewer pointed out this theorem to Davenport.

The form

$$\mathrm{e}^{\mathrm{i}\varphi}uu^\dagger + \mathrm{e}^{-\mathrm{i}\varphi}\tilde u\,\tilde u^\dagger$$

[2]The term ‘sepectral decomposition’ is usually applied to self-adjoint operators, but similar decompositions are possible for a wider class of operators, i.e., that of normal operators, which includes unitary operators.

was used in [6] in order to prove Theorem 1, but there, u is a 'spinor' that is defined in [7], which does not seem widely available. That, in some argument, u may be interpreted as elements in $\mathbb{C}^2$ with operations $u \mapsto \tilde{u}$ and $u \mapsto u^\dagger$ as defined above has been stated by the author of [7] in [13] (his notation for the operations $\tilde{u}$ and $u^\dagger$ is different from ours).

Thus, the expression for rotations in Proposition 1 is close to the form known among those working with rotations [6] in the framework of 'spinor ring algebra' developed in [7]. We also remark that quaternions, rather than Pauli matrices, have been used in [6].

We shall give another form of Theorem 1, which admits a direct comparison with the well-known expression in (5) using the Euler angles. In view of the well-known homomorphism from SU(2) onto SO(3) described in Appendix, we have the following, which is trivially equivalent to Theorem 1.

Theorem 2: [6, Theorem 1]. Let $\hat{n}, \hat{w}, \hat{m} \in S^2$ be given. Any element in SU(2) can be written as $R_{\hat{n}}(\alpha)R_{\hat{w}}(\theta)R_{\hat{m}}(\gamma)$ for some α, θ, γ iff $\hat{w}$ is perpendicular to both $\hat{n}$ and $\hat{m}$.

VI. Conclusion

This article drew the reader's attention to an expression for rotations as SU(2) elements. Relations of this expression to a known expression for rotations as quaternions, which had been written in terms of a little-known algebra, were discussed with emphasis on credit for a known result on decompositons of rotations into triples of rotations.

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Appendix

It is well-known that SU(2) is closely related to SO(3). This appendix gives this relation as a mathematical function, i.e., as a homomorphism from SU(2) onto SO(3).

For $U \in \mathrm{SU}(2)$, we denote by $F(U)$ the matrix of the linear transformation on $\mathbb{R}^3$ that sends $(x,y,z)^{\mathrm{T}}$ to $(x',y',z')^{\mathrm{T}}$ through[3]

$$U(xX + yY + zZ)U^\dagger = x'X + y'Y + z'Z. \tag{9}$$

Namely, for any $(x,y,z)^{\mathrm{T}}, (x',y',z')^{\mathrm{T}} \in \mathbb{R}^3$ with (9),

$$\begin{pmatrix} x' \\ y' \\ z' \end{pmatrix} = F(U) \begin{pmatrix} x \\ y \\ z \end{pmatrix}.$$

We also write

$$\hat{R}_{\hat{v}}(\theta) = F\big(R_{\hat{v}}(\theta)\big), \quad \hat{v} \in S^2, \theta \in \mathbb{R}. \tag{10}$$

This is consistent with the definition of $\hat{R}_{\hat{v}}(\theta)$ in Section II.

[3]Note that in defining the homomorphism in [9], Wigner has used $-Y$ and $-Z$ in place of our Y and Z, which causes a slight difference between his homomorphism and ours, that is, F.

Example. We have

$$\hat{R}_y(\theta) := F(R_y(\theta)) = \begin{pmatrix} \cos\theta & 0 & \sin\theta \\ 0 & 1 & 0 \\ -\sin\theta & 0 & \cos\theta \end{pmatrix} \tag{11}$$

and

$$\hat{R}_z(\theta) := F(R_z(\theta)) = \begin{pmatrix} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{pmatrix}. \tag{12}$$

□

Thus, $\hat{R}_y(\theta)$ and $\hat{R}_z(\theta)$ coincide with the rotations $\hat{R}_{\hat{y}}(\theta)$ and $\hat{R}_{\hat{z}}(\theta)$, respectively, defined in Section II, where $\hat{y} = (0,1,0)^{\mathrm{T}}$ and $\hat{z} = (0,0,1)^{\mathrm{T}}$. It can be seen that $R_{\hat{v}}(\theta)$ with the general $\hat{v} \in S^2$ also represents a rotation (see, e.g., [14] or [2, Section III]), which is consistent with the definition in Section II.

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