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***Abstract*—We calculate the Holevo capacity of attenuation channel with phase sensitive amplifiers and find the optimum arrangement for those amplifiers.**

I. Introduction

The ultimate capability of an optical communication system can be evaluated by computing the Holevo capacity of continuous channel with an energy constraint. Basing on the Holevo's formula[3], [4], we computed it for the attenuation channel assisted by linear amplifiers arranged at equal intervals[7]. Furthermore, we posed the problem of optimum arrangement for phase insensitive amplifiers(PIAs), and found that the position of the last amplifier is crucial while the other amplifiers should be arranged at equal intervals[2]. In this paper we consider the optimum arrangement of phase sensitive amplifiers (PSAs) and find how much capacity is achieved by using them.

II. Attenuation channel with PSAs

Let us start with recalling the description of the Gaussian state based on the quantum characteristic function. We consider a quantum system described by the annihilation operator a satisfying the canonical commutation relation $[a, a^\dagger] = I$, where I is the unit operator and $\dagger$ denotes an adjoint operation, and introduce the canonical pair

$$q = \sqrt{\frac{\hbar}{2}}(a + a^\dagger), \qquad p = i\sqrt{\frac{\hbar}{2}}(a^\dagger - a). \tag{1}$$

Then the quantum characteristic function of a density operator ρ is defined by

$$\chi(z) = \mathrm{Tr}\rho \exp[i(xq + yp)], \tag{2}$$

where z is a two-dimensional vector $z = (x, y)^T$ and T denotes transposition. Note that we confine ourselves to one-mode Gaussian state throughout this paper. In particular the quantum characteristic function of Gaussian state is given by

$$\chi(z) = \exp\left(im^T z - \frac{1}{2} z^T \alpha z\right), \tag{3}$$

with a 2-dimensional mean vector m and a 2×2 correlation matrix α. We denote the Gaussian state with the mean vector m and the correlation matrix α by $\rho(m, \alpha)$. A pure Gaussian state is called a squeezed state which has a correlation matrix of the form

$$\alpha(r) = \frac{\hbar}{2}\begin{pmatrix} e^{-2r} & 0 \\ 0 & e^{2r} \end{pmatrix}$$

where r is a real number, which is called a squeezing parameter. We consider an optical communication system where an attenuation channel with a transmittance $K = k_1 k_2 \cdots k_n k_{n+1}$ is devided into $n+1$ attenuation channels with transmittances $k_i (i = 1, \ldots, n+1)$ and phase sensitive amplifiers(PSAs) with gains $G_i = k_i^{-1} (i = 1, \ldots, n)$ are put (see Figure 1). Such defined optical communication system is called a PSA channel in the following.

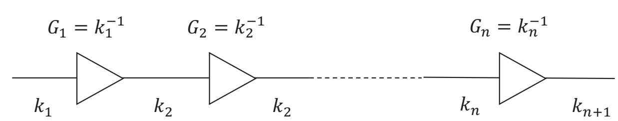

Fig. 1. Schematic diagram of an attenuation channel with PSAs.

The attenuation channel with a transmittance k_i $(0 \le k_i \le 1)$ is described by the transformation

$$a' = k_i^{1/2} a + (1 - k_i)^{1/2} a_0, \tag{4}$$

where a_0 is an annihilation operator in another mode in the Hilbert space $\mathcal{H}_0$ of an "environment", which is initially in the vacuum state[4]. The phase sensitive amplifier (PSA) with a power gain G_i $(G_i > 1)$ is an ideal parametric amplifier represented by

$$q' = G_i^{1/2} q, \quad p' = G_i^{-1/2} p$$

where the q is maximally amplified while the quadrature p is correspondingly maximally attenuated [5]. The change of Gaussian state caused by attenuation or phase sensitive amplification can be described in terms of the mean vector and the correlation matrix. Let us consider a Gaussian state with a mean vector $m = (m_q, m_p)^T$ and a correlation matrix

$$\alpha = \begin{pmatrix} \alpha_{qq} & 0 \\ 0 & \alpha_{pp} \end{pmatrix}.$$

Then the output state from the attenuation channel with a transmittance k_i has the mean vector

$$(k_i^{1/2} m_q, k_i^{1/2} m_p)^T$$

and the correlation matrix

$$\psi_{k_i}(\alpha) = k_i\alpha + \hbar\frac{1-k_i}{2}I_2,$$

with the 2×2 identity matrix I_2. On the other hand the output state from the PSA with a power gain G_i has the mean vector

$$(G_i^{1/2}m_q, G_i^{-1/2}m_p)^T$$

and the correlation matrix

$$\phi_{G_i}(\alpha) = \begin{pmatrix} G_i\alpha_{qq} & 0 \\ 0 & G_i^{-1}\alpha_{pp}\end{pmatrix}.$$

Now we are ready to obtain the mean vector m' and the correlation matrix α' of the output Gaussian state $\Phi(\rho(m,\alpha(r)))$ from the PSA channel. The mean vector $m' = (m'_q, m'_p)^T$ is given by

$$\begin{aligned} m'_q =& k_1^{1/2}G_1^{1/2}\cdots k_n^{1/2}G_n^{1/2}k_{n+1}^{1/2}m_q = k_{n+1}^{1/2}m_q \\ m'_p =& k_1^{1/2}G_1^{-1/2}\cdots k_n^{1/2}G_n^{-1/2}k_{n+1}^{1/2}m_p \\ =& k_1\cdots k_n k_{n+1}^{1/2}m_p \\ =& K/k_{n+1}^{1/2}m_p. \end{aligned} \tag{5}$$

In other words m and m' are related by the equation

$$m' = L_\Phi m$$

with

$$L_\Phi = \begin{pmatrix} k_{n+1}^{1/2} & 0 \\ 0 & K/k_{n+1}^{1/2}\end{pmatrix}.$$

The correlation matrix α' can be obtained recursively. Let the correlation matrix of output Gaussian state from the i-th PSA

$$\alpha_i(r) = \frac{\hbar}{2}\begin{pmatrix} a_i & 0 \\ 0 & b_i\end{pmatrix}, \quad i = 1,2,...,n,$$

and the correlation matrix of the input squeeze state

$$\alpha_0(r) = \frac{\hbar}{2}\begin{pmatrix} a_0 & 0 \\ 0 & b_0\end{pmatrix}$$

with $a_0 = e^{-2r}, b_0 = e^{2r}$. Then we have

$$\begin{aligned} a_i &= G_i(k_i a_{i-1} + 1 - k_i) = a_{i-1} + k_i^{-1} - 1, \\ b_i &= G_i^{-1}(k_i b_{i-1} + 1 - k_i) = k_i^2 b_{i-1} + k_i - k_i^2, \end{aligned} \tag{6}$$

with $i = 1,2,...,n$. The correlation matrix

$$\alpha' = \frac{\hbar}{2}\begin{pmatrix} a & 0 \\ 0 & b\end{pmatrix}$$

of the output Gaussian state from the PSA channel is given by

$$\psi_{k_{n+1}}(\alpha_n(r)).$$

Computing the values of a_n and b_n through the recurrence relations (6), we obtain the elements of α' as

$$\begin{aligned} a &= k_{n+1}e^{-2r} + t_a \\ b &= K^2/k_{n+1}e^{2r} + t_b \end{aligned} \tag{7}$$

with

$$t_a = k_{n+1}\sum_{i=1}^n k_i^{-1} - nk_{n+1} + 1 - k_{n+1}.$$

Here t_b is a polynomial function of $k_1, ..., k_n, k_{n+1}$, which can be obtained ony recursively. Thus we have found the way to compute the mean vector m' and the correlation matrix α' of the output Gaussian state $\Phi(\rho(m,\alpha(r)))$.

III. Computation of the Holevo capacity of the PSA channel

We now pass to the computation of the Holevo capacity. The capacity for the transmission of classical information with squeezed states $\rho(m,\alpha(r))$ through the PSA channel Φ is given as

$$\begin{aligned} C(r) = \sup_\pi \Bigg[& S\left(\int \Phi(\rho(m,\alpha(r))\pi(dm)\right) \\ & - \int S\left(\Phi(\rho(m,\alpha(r)))\,\pi(dm)\right)\Bigg], \end{aligned} \tag{8}$$

by using the Holevo's capacity formula[6]. Here S is the von Neumann entropy $S(\rho) = -\mathrm{Tr}\rho\log\rho$ and π is an *apriori* probability distribution subject to the energy constraint

$$\mathrm{Tr}H\int \rho(m,\alpha(r))\pi(dm) \le \hbar(N_{tr} + 1/2), \tag{9}$$

where $H = \hbar(a^\dagger a + 1/2)$ and N_{tr} is a constant representing an average photon number. In order to compute the Holevo capacity $C(r)$, let us recall firstly that the von Neumann entropy for a one-mode Gaussian state with a correlation matrix γ is given by

$$g\left(\frac{\sqrt{\det\gamma}}{\hbar} - \frac{1}{2}\right), \tag{10}$$

with $g(x) = (x+1)\log(x+1) - x\log x$. We assume that the *a priori* probability distribution $\pi(dm)$ in Eq. (8) can be restricted to a classical Gaussian distribution with the mean 0 and a diagonal correlation matrix β. Then the correlation matrix β' for the distribution of mean vector of output Gaussian state $\Phi(\rho(m,\alpha(r)))$ is given by

$$\beta' = L_\Phi\beta L_\Phi^T, \tag{11}$$

and we can rewrite Eq. (8) and Eq. (9) as

$$C(r) = \max_\beta g\left(\frac{\sqrt{\det(\alpha'+\beta')}}{\hbar} - \frac{1}{2}\right) - g\left(\frac{\sqrt{\det\alpha'}}{\hbar} - \frac{1}{2}\right) \tag{12}$$

and

$$\frac{1}{2}\mathrm{Sp}\alpha(r) + \frac{1}{2}\mathrm{Sp}\beta \le \hbar\left(N_{tr} + \frac{1}{2}\right), \tag{13}$$

where "Sp" denotes trace of matrix as distinct from trace of operator "Tr". Since $\mathrm{Sp}\beta \ge 0$ holds in Eq. (13), we have $\mathrm{Sp}\alpha(r) \le \hbar(2N_{tr}+1)$, which leads to

$$r_- \le r \le r_+$$

with

$$r_\pm = \frac{1}{2}\log[2N_{tr} + 1 \pm 2\sqrt{N_{tr}(N_{tr}+1)}].$$

Here it is easy to show that the equality in Eq. (13) should hold for the optimum correlation matrix β, and hence we use the energy constraint

$$\frac{1}{2}\mathrm{Sp}\alpha(r) + \frac{1}{2}\mathrm{Sp}\beta = \hbar\left(N_{tr} + \frac{1}{2}\right)$$

in place of Eq. (13). Using Eq. (13) and Eq. (11) we obtain the constraint for the correlation matrix of output Gaussian state

$$\beta' = \frac{\hbar}{2}\begin{pmatrix} c & 0 \\ 0 & d \end{pmatrix}$$

as

$$\frac{\hbar}{4}\left(\frac{c}{k_{n+1}} + \frac{k_{n+1}d}{K^2}\right) = \hbar N_s,$$

with

$$N_s = N_{tr} + \frac{1}{2} - \frac{1}{2}\cosh 2r$$

that is,

$$c = 4k_{n+1}N_s - \frac{k_{n+1}^2}{K^2}d. \tag{14}$$

Here from the positivity of β', we have

$$0 \le d \le \frac{4K^2}{k_{n+1}}N_s. \tag{15}$$

Since the second term in Eq. (12) is a constant and $g(\sqrt{x} - 1/2)$ is an increasing function, we can obtain the capacity by maximizing

$$\frac{det(\alpha' + \beta')}{\hbar^2} = \frac{1}{4}(a+c)(b+d) \tag{16}$$

under the conditions Eq. (14) and Eq. (15). Substituting Eq. (14) into the right-hand side of Eq.(16), we obtain a quadratic function $Q(d)$, whose axis of symmetry is $d = \ell$ with

$$\ell = \frac{K^2}{k_{n+1}}\left(2N_s + \frac{1}{2k_{n+1}}a - \frac{k_{n+1}}{2K^2}b\right).$$

The maximum value of $Q(d)$ under the constraint (15) is given as follows.

(i) When $\ell < 0$, that is,

$$a < \frac{k_{n+1}^2}{K^2}b - 4k_{n+1}N_s,$$

the maximum value of $Q(d)$ is achieved as

$$Q_{max} = \frac{1}{4}(a + 4k_{n+1}N_s)b$$

by

$$\begin{aligned} d &= 0 \\ c &= 4k_{n+1}N_s. \end{aligned} \tag{17}$$

(ii) When $\ell > 4K^2N_s/k_{n+1}$, that is,

$$a > \frac{k_{n+1}^2}{K^2}b + 4k_{n+1}N_s,$$

the maximum value of $Q(d)$ is achieved as

$$Q_{max} = \frac{1}{4}a\left(b + \frac{4K^2}{k_{n+1}}N_s\right)$$

by

$$\begin{aligned} c &= 0 \\ d &= \frac{4K^2}{k_{n+1}}N_s. \end{aligned} \tag{18}$$

(iii) When $0 \le \ell \le 4K^2N_s/k_{n+1}$, the maximum value of $Q(d)$ is achieved as

$$Q_{max} = K^2\left(N_s + \frac{1}{4k_{n+1}}a + \frac{k_{n+1}}{4K^2}b\right)^2$$

by

$$\begin{aligned} c &= k_{n+1}\left(2N_s - \frac{1}{2k_{n+1}}a + \frac{k_{n+1}}{2K^2}b\right) \\ d &= \frac{K^2}{k_{n+1}}\left(2N_s + \frac{1}{2k_{n+1}}a - \frac{k_{n+1}}{2K^2}b\right). \end{aligned} \tag{19}$$

Thus we can obtain the Holevo capacity as

$$C(r) = g\left(\sqrt{Q_{max}} - \frac{1}{2}\right) - g\left(\sqrt{ab} - \frac{1}{2}\right).$$

Moreover we maximize $C(r)$ with respect to arrangement of PSAs determined by $k_1, ..., k_{n+1}$ to obtain the Holevo capacity

$$C_{opt}(r) = \max_{k_1,...,k_{n+1}} C(r),$$

where transmitances $k_1, ..., k_{n+1}$ satify $0 < k_i \le 1$ $(i = 1, ..., n+1)$ and $K = k_1 \cdots k_{n+1}$ for the total transmittance K.

Let us evaluate the capacity of the PSA channel. Figure 2 shows graphs of Holevo capacities $C_{opt}(r)$ of attenuation channels with n PSAs ($n = 1, 5, 10$) and an attenuation channel with no amplifier when $K = 0.1$ and $N_{tr} = 1$. In the figure the horizontal axis represents the squeezing parameter r. These graphs show that the PSA channel requires the squeezed state as the input state, while squeezing is not useful for the attenuation channel with no amplifier.

Next we consider arrangement of PSAs. For that we compare $C_{opt}(r)$ with the Holevo capacity $C_{eq}(r)$ which is obtained for PSAs arranged at equal intervals, that is, $k_1 = \cdots = k_n = k_{n+1}$. Moreover we maximize them with respect to the squeezing parameter r to get $C_{opt} = \max_r C_{opt}(r)$ and $C_{eq} = \max_r C_{eq}(r)$. Figure 3 shows graphs of the Holevo capacities C_{opt} and C_{eq} with respect to the number of amplifiers when $N_{tr} = 1$ and $K = 0.1$. It indicates that arranging amplifiers at equal intervals is not the best way to enhance the Holevo capacity.

Figure 4 shows the best arrangement of PSAs when $N_{tr} = 1$ and $K = 0.1$. In the figure the horizontal axis represents the index i of sub-channel and the vertical axis transmission loss of sub-channel $10\log_{10} 1/k_i$. From the graphs we can see that in the optimum arrangement length

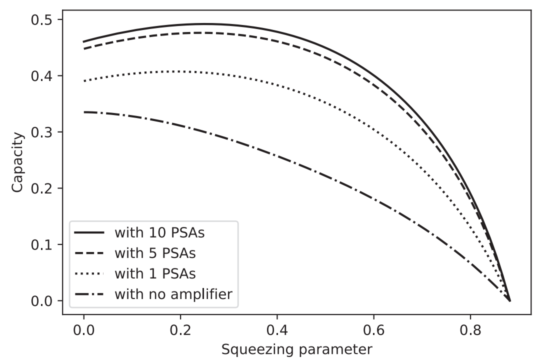

Fig. 2. Dependence of Holevo capacities for attenuation channels with/without phase sensitive amplifiers on the squeezing parameter r.

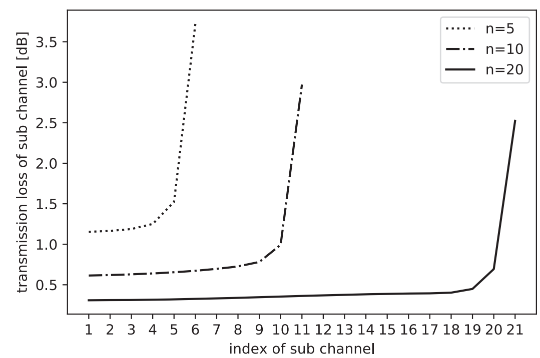

Fig. 4. Optimum arrangement described by values of transmission loss of sub-channel $10 \log 1/k_i$,$i = 1, ..., n+1$ for $n = 5, 10, 20$.

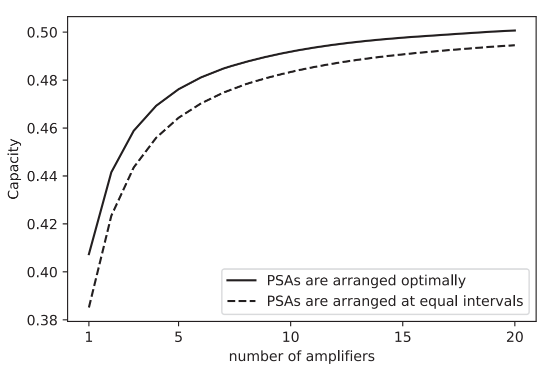

Fig. 3. Dependance of Holevo capacities for attenumation channels on the number of amplifiers n when PSAs are arranged optimally or equally.

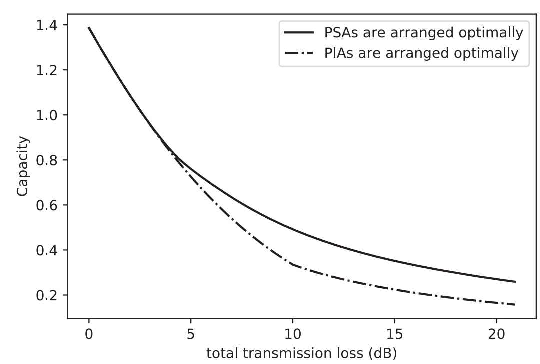

Fig. 5. Dependance of Holevo capacities for attenumation channels with PIAs/PSAs on the total transmission loss when $N_{tr} = 1$ and $n = 10$.

of several sub-channels near the receiver are longer than the others.

Finally we compare the effect of PSA with that of phase insensitive amplifier (PIA). The PIA is a usual linear amplifier, which is described by the transformation in the Heisenberg picture,

$$a' = G_i^{1/2} a + (G_i - 1)^{1/2} a_0^\dagger, \tag{20}$$

where a (resp. a') is an input modal photon annihilation operator (resp. an output one), $a_0^\dagger$ is a creation operator representing the additive noise introduced by the amplification, and $G_i (\geq 1)$ denotes a power gain of amplifier. We proved [2] that $k_1 = k_2 = \cdots = k_n = (K/k_{n+1})^{1/n}$ holds in the optimum arangement of PIAs and that for a fixed $k_{n+1} = k$ the maximum value of the Holevo capacity is given by

$$C_n(k) = g(kN_{tr}, N_n) \tag{21}$$

with

$$\begin{aligned} g(E, N) =& \log\left(1 + \frac{E}{N+1}\right) + (N+E)\log\left(1 + \frac{1}{N+E}\right) \\ & - N \log\left(1 + \frac{1}{N}\right) \\ N_n =& k \cdot n \left(\left(\frac{k}{K}\right)^{1/n} - 1\right). \end{aligned} \tag{22}$$

We maximize $C_n(k)$ numerically with respect to k to obtain the Holevo capacity C'_{opt} achieved by the optimum arrangement of PIAs. Figure 5 and Figure 6 show graphs of the capacities C_{opt} and C'_{opt} with respect to the total transmission loss $\log_{10} 1/K$ for $N_{tr} = 1$ and $N_{tr} = 20$ respectively. These graphs indicate that PSAs enhance the Holevo capacity more than PIAs when power of input signals are small or total transmittance is small.

IV. Conclusion

We have computed the Holevo capacity of PSA channel and the optimum arrangement of PSAs. We will study

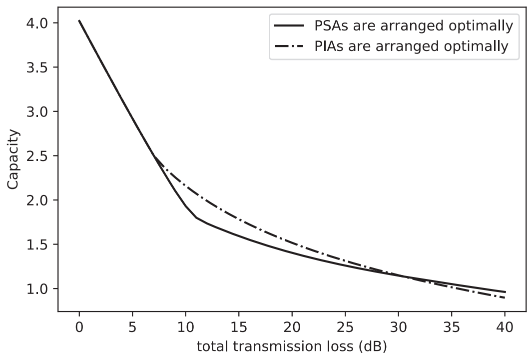

Fig. 6. Dependance of Holevo capacities for attenumation channels with PIAs/PSAs on the total transmission loss when $N_{tr} = 20$ and $n = 10$.

on optimum amplifiers and their arrangement in a more general setting.

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