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A Note on the Error Probability by Homodyne Receiver for *M*-ary PSK Coherent State Signal via Optical Transmission Lines with Amplifiers

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***Abstract*—An approximation formula of the error probability of M-ary PSK coherent state signal by a homodyne receiver is considered in the case that the communication channel contains in-line erbium amplifiers. Applying Cahn's calculation method [1] to our problem, an approximation formula for M-ary PSK coherent state signal in that case is obtained.**

I. Introduction

In the literature [2], phase shift keying (PSK) and quadrature amplitude modulation (QAM) coherent state signals were investigated in terms of quantum signal detection theory, and the error probability and the mutual information that are achieved by optimal quantum receivers and ideal homodyne receivers were computed for each coherent state signal. As for the case of the homodyne receiver for PSK coherent state signals, all the calculation was executed by numerical integral, because it is difficult to find the closed-form expression of the error probability by the homodyne receiver except for the cases of binary PSK (BPSK) and quaternary PSK (QPSK) coherent state signals. However, a useful approximation formula of the error probability of M-ary PSK signal has been already obtained by Cahn in the context of radio wave communications [1].

In this article, an approximation formula of the error probability of M-ary PSK coherent state signal by a two-quadrature field measurement type homodyne receiver is considered in the case that the communication channel contains in-line erbium amplifiers. First, we summarize quantum mechanical treatment of optical transmission lines with in-line erbium amplifiers according to the literature [3]. It is well known that a two-quadrature field measurement type homodyne receiver is realized by an eight-port homodyne detector (*e.g.*, [4]). To apply this result to our case, we give a simple analysis of the eight-port homodyne detector by means of the Skellam distribution [5]. Finally, we combine these two quantum mechanical treatments of the channel and detector. Based on this, an approximation formula of the error probability of M-ary coherent state signal by the homodyne receiver is derived by means of Cahn's calculation method.

II. Channel Model

In this article, we employ Mecozzi's model of N loss-gain optical transmission lines with in-line erbium amplifiers [3]. Let Γ_k denote the loss of the kth segment and G_k the gain of the kth segment, where $1 \geq \Gamma_k > 0$ and $G_k \geq 1$. The spontaneous emission factor of the kth in-line erbium amplifier is denoted by $n_{k,\mathrm{sp}} \geq 1$. A schematic of this channel is shown in Fig. 1.

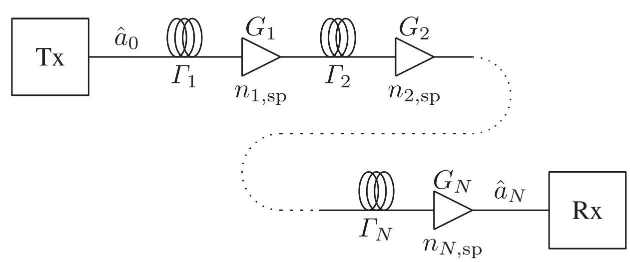

Fig. 1. Channel model of N loss-gain optical transmission lines [3]

The mode of the input to this channel is expressed by the photon annihilation operator $\hat{a}_0$, *i.e.*, $[\hat{a}_0, \hat{a}_0^\dagger] = \hat{1}$. According to the literature [3], the output mode $\hat{a}_N$ is given by

$$\begin{aligned}\hat{a}_N &= \sqrt{G_{\mathrm{net}}}\,\hat{a}_0 \\ &\quad + \sqrt{n_{\mathrm{ASE}} - (G_{\mathrm{net}} - 1)}\,\hat{g}_1 + \sqrt{n_{\mathrm{ASE}}}\,\hat{g}_2^\dagger.\end{aligned}$$

Here the net gain is defined by $G_{\mathrm{net}} = \prod_{j=1}^{N} \Gamma_j G_j$ and the total number of amplified spontaneous emission (ASE) noise photons is $n_{\mathrm{ASE}} = \sum_{k=1}^{N} f(N, k+1) n_{k,\mathrm{sp}}(G_k - 1)$, in which the function f is defined by $f(k,h) \equiv \prod_{j=h}^{k} \Gamma_j G_j$ with convention $f(N, N+1) \equiv 1$.

Suppose the mode $\hat{a}_0$ is in a coherent state $|\alpha\rangle$ having complex amplitude $\alpha = A_I + \mathrm{i}A_Q \in \mathbb{C}$ ($A_I = \mathrm{Re}[\alpha]$, $A_Q = \mathrm{Im}[\alpha]$, and $\mathrm{i} = \sqrt{-1}$) and the other two modes $\hat{g}_1$ and $\hat{g}_2^\dagger$, which respectively correspond to the total absorption and spontaneous emission processes, are in the vacuum states. So, we let $|\Psi\rangle \equiv |\alpha\rangle_0 \otimes |0\rangle_1 \otimes |0\rangle_2$. Straightforward calculations yield the following results:

$$\langle \hat{a}_N \rangle = \langle \Psi | \hat{a}_N | \Psi \rangle = \sqrt{G_{\mathrm{net}}}\,\alpha, \tag{1}$$

$$\langle \hat{a}_N^\dagger \rangle = \langle \Psi | \hat{a}_N^\dagger | \Psi \rangle = \sqrt{G_{\mathrm{net}}}\,\alpha^*, \tag{2}$$

where $*$ stands for the complex conjugate,

$$\langle \hat{a}_N^2 \rangle = \langle \Psi | \hat{a}_N^2 | \Psi \rangle = G_{\text{net}} \alpha^2, \quad (3)$$

$$\langle \hat{a}_N^{\dagger 2} \rangle = \langle \Psi | \hat{a}_N^{\dagger 2} | \Psi \rangle = G_{\text{net}} \alpha^{*2}, \quad (4)$$

and

$$\langle \hat{a}_N^\dagger \hat{a}_N \rangle = \langle \Psi | \hat{a}_N^\dagger \hat{a}_N | \Psi \rangle = G_{\text{net}} |\alpha|^2 + n_{\text{ASE}}, \quad (5)$$

$$\langle \hat{a}_N \hat{a}_N^\dagger \rangle = \langle \Psi | \hat{a}_N \hat{a}_N^\dagger | \Psi \rangle = G_{\text{net}} |\alpha|^2 + n_{\text{ASE}} + 1. \quad (6)$$

Further, the quantum state at the output mode $\hat{a}_N$ for coherent state input $|\alpha\rangle$ at the input mode $\hat{a}_0$ is given by

$$\hat{\rho}(\alpha) = \frac{1}{\pi n_{\text{ASE}}} \int \mathrm{d}^2\eta \exp[-\frac{|\eta - \sqrt{G_{\text{net}}}\ \alpha|^2}{n_{\text{ASE}}}] |\eta\rangle\langle\eta|, \quad (7)$$

where $\mathrm{d}^2\eta \equiv \mathrm{d}(\mathrm{Re}[\eta])\mathrm{d}(\mathrm{Im}[\eta])$ and $|\eta\rangle$ is a coherent state having complex amplitude η.

Let $|\zeta\rangle$ be a coherent state having complex amplitude $\zeta = \zeta_I + \mathrm{i}\zeta_Q$ ($\zeta_I = \mathrm{Re}[\zeta]$ and $\zeta_Q = \mathrm{Im}[\zeta]$). Because of the overcompleteness property of the coherent states, the collection $\{(1/\pi)|\zeta\rangle\langle\zeta| : \zeta \in \mathbb{C}\}$ becomes a positive operator-valued measure (POVM). This corresponds to two-quadrature field measurement. The probability density function of the outcome ζ by this POVM is

$$p(\zeta|\alpha) = \frac{1}{\pi}\langle\zeta|\hat{\rho}(\alpha)|\zeta\rangle = \frac{1}{\pi(n_{\text{ASE}}+1)} \exp[-\frac{|\zeta - \sqrt{G_{\text{net}}}\ \alpha|^2}{n_{\text{ASE}}+1}]. \quad (8)$$

Substituting $\zeta = \zeta_I + \mathrm{i}\zeta_Q$ and $\alpha = A_I + \mathrm{i}A_Q$, this probability density function can be rewritten as

$$p(\zeta_I, \zeta_Q | A_I, A_Q) = \frac{1}{\sqrt{\pi(n_{\text{ASE}}+1)}} \exp[-\frac{(\zeta_I - \sqrt{G_{\text{net}}}\ A_I)^2}{n_{\text{ASE}}+1}] \times \frac{1}{\sqrt{\pi(n_{\text{ASE}}+1)}} \exp[-\frac{(\zeta_Q - \sqrt{G_{\text{net}}}\ A_Q)^2}{n_{\text{ASE}}+1}]. \quad (9)$$

Hence the expected values of the outcome by this two-quadrature measurement are

$$\mu_I^{\text{POVM}} = \sqrt{G_{\text{net}}}\ A_I, \quad \mu_Q^{\text{POVM}} = \sqrt{G_{\text{net}}}\ A_Q, \quad (10)$$

and the variances are

$$(\sigma_I^2)^{\text{POVM}} = (\sigma_Q^2)^{\text{POVM}} = \frac{n_{\text{ASE}}+1}{2}. \quad (11)$$

III. Eight-port Homodyne Detector

Before calculating the error probability of M-ary PSK coherent sate signal, we revisit a quantum theory of an eight-port homodyne detector (*e.g.*, [4], [8], [9], [10], [11], [12], [13], [14], [15]). From the preceding studies on the eight-port homodyne detector, the probability density function (8), or (9), has been justified in various ways.

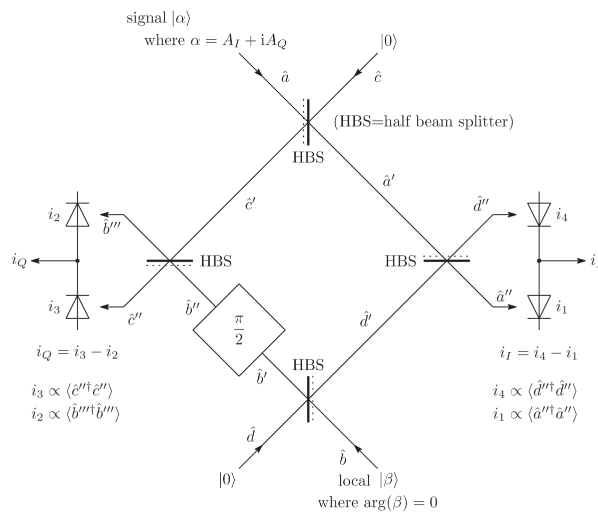

Fig. 2. Eight-port homodyne detector

In this section, this statistical property of the eight-port homodyne detector is restated by means of the *Skellam distribution* (See Appendix A).

A schematic of the detector is shown in Fig. 2. This consists of four half beam splitters (HBSs), one $\pi/2$-phase shifter, and four photodetectors. The input mode $\hat{a}$ is used for receiving signals and the input mode $\hat{b}$ is for the local oscillator. The remaining two input modes $\hat{c}$ and $\hat{d}$ are in the vacuum.

In each HBS, the corresponding mode operators satisfy the following relations [6]:

$$\hat{a}' = \frac{1}{\sqrt{2}}(\hat{a} - \hat{c}) \quad \text{and} \quad \hat{c}' = \frac{1}{\sqrt{2}}(\hat{a} + \hat{c}); \quad (12)$$

$$\hat{b}' = \frac{1}{\sqrt{2}}(\hat{b} - \hat{d}) \quad \text{and} \quad \hat{d}' = \frac{1}{\sqrt{2}}(\hat{b} + \hat{d}); \quad (13)$$

$$\hat{a}'' = \frac{1}{\sqrt{2}}(\hat{a}' - \hat{d}') \quad \text{and} \quad \hat{d}'' = \frac{1}{\sqrt{2}}(\hat{a}' + \hat{d}'); \quad (14)$$

$$\hat{b}''' = \frac{1}{\sqrt{2}}(\hat{b}'' - \hat{c}') \quad \text{and} \quad \hat{c}'' = \frac{1}{\sqrt{2}}(\hat{b}'' + \hat{c}'), \quad (15)$$

where $\hat{b}'' = \mathrm{e}^{\mathrm{i}\pi/2}\hat{b}' = \mathrm{i}\hat{b}'$. With a small algebra, we obtain the following expressions of the output modes from the HBSs.

$$\hat{a}'' = \frac{1}{2}(\hat{a} - \hat{b} - \hat{c} - \hat{d}); \quad (16)$$

$$\hat{b}''' = \frac{1}{2}(-\hat{a} + \mathrm{i}\hat{b} - \hat{c} - \mathrm{i}\hat{d}); \quad (17)$$

$$\hat{c}'' = \frac{1}{2}(\hat{a} + \mathrm{i}\hat{b} + \hat{c} - \mathrm{i}\hat{d}); \quad (18)$$

$$\hat{d}'' = \frac{1}{2}(\hat{a} + \hat{b} - \hat{c} + \hat{d}). \quad (19)$$

Here we define $\hat{i}_1 = \hat{a}''^\dagger \hat{a}''$, $\hat{i}_2 = \hat{b}'''^\dagger \hat{b}'''$, $\hat{i}_3 = \hat{c}''^\dagger \hat{c}''$, and $\hat{i}_4 = \hat{d}''^\dagger \hat{d}''$. The final output modes $\hat{i}_I = \hat{i}_4 - \hat{i}_1$ and

$\hat{i}_Q = \hat{i}_3 - \hat{i}_2$ are respectively expressed as

$$\hat{i}_I = \frac{1}{2}\Big(\hat{a}^\dagger\hat{b} + \hat{b}^\dagger\hat{a} + \hat{a}^\dagger\hat{d} + \hat{d}^\dagger\hat{a} - \hat{b}^\dagger\hat{c} - \hat{c}^\dagger\hat{b} - \hat{c}^\dagger\hat{d} - \hat{d}^\dagger\hat{c}\Big) \quad (20)$$

and

$$\hat{i}_Q = \frac{\mathrm{i}}{2}\Big(\hat{a}^\dagger\hat{b} - \hat{b}^\dagger\hat{a} - \hat{a}^\dagger\hat{d} + \hat{d}^\dagger\hat{a} - \hat{b}^\dagger\hat{c} + \hat{c}^\dagger\hat{b} - \hat{c}^\dagger\hat{d} + \hat{d}^\dagger\hat{c}\Big). \quad (21)$$

Further we obtain

$$\hat{i}_I^2 = \frac{1}{4}\Big(\hat{a}^{\dagger 2}\hat{b}^2 + \hat{a}^2\hat{b}^{\dagger 2} + 2\hat{a}^\dagger\hat{a} + 2\hat{a}^\dagger\hat{a}\hat{b}^\dagger\hat{b} + 2\hat{b}^\dagger\hat{b} + \hat{R}_1\Big) \quad (22)$$

and

$$\hat{i}_Q^2 = -\frac{1}{4}\Big(\hat{a}^{\dagger 2}\hat{b}^2 + \hat{a}^2\hat{b}^{\dagger 2} - 2\hat{a}^\dagger\hat{a} - 2\hat{a}^\dagger\hat{a}\hat{b}^\dagger\hat{b} - 2\hat{b}^\dagger\hat{b} + \hat{R}_2\Big), \quad (23)$$

where $\hat{R}_1$ and $\hat{R}_2$ are the remaining terms that vanish when modes $\hat{c}$ and $\hat{d}$ are in the vacuum.

A. *Case of Coherent State*

Suppose the input signal is a coherent state $|\alpha = A_I + \mathrm{i}A_Q\rangle$ and the local oscillator light is a coherent state $|\beta\rangle$ of $\beta > 0$. That is, the whole input state of the detector is $|\mathbf{\Phi}\rangle_{abcd}^{\mathrm{in}} = |\alpha\rangle_a \otimes |\beta\rangle_b \otimes |0\rangle_c \otimes |0\rangle_d$.

From Eqs. (20) and (21), the expected values of the outcome by this detector are

$$\langle\hat{i}_I\rangle = \langle\hat{i}_4 - \hat{i}_1\rangle = A_I\beta, \quad (24)$$

$$\langle\hat{i}_Q\rangle = \langle\hat{i}_3 - \hat{i}_2\rangle = A_Q\beta. \quad (25)$$

By using Eqs. (22) and (23) together with the results above, the corresponding variances are calculated as

$$\langle(\varDelta\hat{i}_I)^2\rangle = \langle\hat{i}_I^2\rangle - \langle\hat{i}_I\rangle^2 = \frac{1}{2}(|\alpha|^2 + \beta^2), \quad (26)$$

$$\langle(\varDelta\hat{i}_Q)^2\rangle = \langle\hat{i}_Q^2\rangle - \langle\hat{i}_Q\rangle^2 = \frac{1}{2}(|\alpha|^2 + \beta^2). \quad (27)$$

Since the input state $|\mathbf{\Phi}\rangle_{abcd}^{\mathrm{in}}$ contains only coherent states and the vacuum, the output state is obtained as

$$|\mathbf{\Phi}\rangle_{abcd}^{\mathrm{out}} = |\frac{1}{2}(\alpha - \beta)\rangle_a \otimes |\frac{1}{2}(-\alpha + \mathrm{i}\beta)\rangle_b \otimes |\frac{1}{2}(\alpha + \mathrm{i}\beta)\rangle_c \otimes |\frac{1}{2}(\alpha + \beta)\rangle_d. \quad (28)$$

From this, the corresponding photon statistics are summarized as follows.

- mode $\hat{a}''$: Poisson with parameter $|\alpha - \beta|^2/4$.
- mode $\hat{b}'''$: Poisson with parameter $|\alpha - \mathrm{i}\beta|^2/4$.
- mode $\hat{c}''$: Poisson with parameter $|\alpha + \mathrm{i}\beta|^2/4$.
- mode $\hat{d}''$: Poisson with parameter $|\alpha + \beta|^2/4$.

A distribution of the difference of two Poisson random valuables is known as the *Skellam distribution*. From Eqs. (71) and (72) of Appendix A, the expected value μ_I and the variance σ_I^2 of the mode $\hat{i}_I$ are respectively given as

$$\mu_I = \frac{1}{4}|\alpha + \beta|^2 - \frac{1}{4}|\alpha - \beta|^2 = A_I\beta \quad (29)$$

and

$$\sigma_I^2 = \frac{1}{4}|\alpha + \beta|^2 + \frac{1}{4}|\alpha - \beta|^2 = \frac{1}{2}(|\alpha|^2 + \beta^2), \quad (30)$$

which are identical to Eqs. (24) and (26). Substituting these parameters into Eq. (70) of Appendix A, the probability mass function of the number n_I of output photon is

$$P_I(n_I) = \exp[-\frac{1}{2}(|\alpha|^2 + \beta^2)] \times \left|\frac{\alpha + \beta}{\alpha - \beta}\right|^{n_I} \mathsf{I}_{n_I}[\frac{1}{2}|\alpha^2 - \beta^2|]. \quad (31)$$

Similarly, the mode $\hat{i}_Q$ has the probability mass function of the number n_Q of output photon,

$$P_Q(n_Q) = \exp[-\frac{1}{2}(|\alpha|^2 + \beta^2)] \times \left|\frac{\alpha + \mathrm{i}\beta}{\alpha - \mathrm{i}\beta}\right|^{n_Q} \mathsf{I}_{n_Q}[\frac{1}{2}|\alpha^2 + \beta^2|], \quad (32)$$

and its expected value μ_Q and variance σ_Q^2 are respectively given by

$$\mu_Q = \frac{1}{4}|\alpha + \mathrm{i}\beta|^2 - \frac{1}{4}|-\alpha + \mathrm{i}\beta|^2 = A_Q\beta, \quad (33)$$

$$\sigma_Q^2 = \frac{1}{4}|\alpha + \mathrm{i}\beta|^2 + \frac{1}{4}|-\alpha + \mathrm{i}\beta|^2 = \frac{1}{2}(|\alpha|^2 + \beta^2). \quad (34)$$

Here we assume $\beta \gg |\alpha| \gg 1$, which means the use of strong local oscillator. From Eq. (76) of Appendix A. we have

$$P_I(n_I) \approx \frac{1}{\sqrt{\pi\beta^2}}\exp[-\frac{|n_I - A_I\beta|^2}{\beta^2}] \quad (35)$$

with expected value $A_I\beta$ and variance $\beta^2/2$, and

$$P_Q(n_Q) \approx \frac{1}{\sqrt{\pi\beta^2}}\exp[-\frac{|n_Q - A_Q\beta|^2}{\beta^2}] \quad (36)$$

with expected value $A_Q\beta$ and variance $\beta^2/2$.

For rescaling measurement outcomes by β, we define $z_I = n_I/\beta$ and $z_Q = n_Q/\beta$. Since β is large enough, z_I and z_Q can be regarded as real numbers within a finite precision. So, we now reached to the probability density functions

$$p_I(z_I)\mathrm{d}z_I \approx \frac{1}{\sqrt{\pi}}\exp[-|z_I - A_I|^2]\mathrm{d}z_I \quad (37)$$

with expected value A_I and variance $1/2$, and

$$p_Q(z_Q)\mathrm{d}z_Q \approx \frac{1}{\sqrt{\pi}}\exp[-|z_Q - A_Q|^2]\mathrm{d}z_Q \quad (38)$$

with expected value A_Q and variance $1/2$. Defining $z = z_I + \mathrm{i} z_Q$, the joint distribution of (z_I, z_Q) is given by

$$\begin{aligned} p_{IQ}(z_I, z_Q|\alpha) &= p_I(z_I) p_Q(z_Q) \\ &\approx \frac{1}{\pi}\exp[-|z-\alpha|^2] \\ &= \frac{1}{\pi}|\langle z|\alpha\rangle|^2. \end{aligned} \tag{39}$$

This corresponds to a special case of Eq. (8) when $G_{\text{net}} = 1$ and $n_{\text{ASE}} = 0$.

B. Case of the state of Eq. (7)

Suppose the communication channel of N loss-gain optical transmission lines is connected to the detector. That is, we assume $\hat{a} = \hat{a}_N$. When the transmission input mode $\hat{a}_0$ is in a coherent state $|\alpha = A_I + \mathrm{i}A_Q\rangle$, the expected values and variances of the final outputs are given as follows: the expected values of $\hat{i}_I$ and $\hat{i}_Q$ are

$$\langle \hat{i}_I\rangle = \sqrt{G_{\text{net}}}A_I\beta, \tag{40}$$
$$\langle \hat{i}_Q\rangle = \sqrt{G_{\text{net}}}A_Q\beta \tag{41}$$

and the variances of $\hat{i}_I$ and $\hat{i}_Q$ are

$$\begin{aligned} \langle(\Delta\hat{i}_I)^2\rangle &= \frac{1}{2}\left\{G_{\text{net}}|\alpha|^2 + n_{\text{ASE}} + (n_{\text{ASE}}+1)\beta^2\right\} \\ &\approx \frac{1}{2}(n_{\text{ASE}}+1)\beta^2, \end{aligned} \tag{42}$$
$$\begin{aligned} \langle(\Delta\hat{i}_Q)^2\rangle &= \frac{1}{2}\left\{G_{\text{net}}|\alpha|^2 + n_{\text{ASE}} + (n_{\text{ASE}}+1)\beta^2\right\} \\ &\approx \frac{1}{2}(n_{\text{ASE}}+1)\beta^2, \end{aligned} \tag{43}$$

where we have assumed $\beta \gg \sqrt{G_{\text{net}}}|\alpha|$ to obtain the approximations above.

To see the distribution of the final output, we let

$$\mathsf{P}(\eta) = \frac{1}{\pi n_{\text{ASE}}}\exp[-\frac{|\eta-\sqrt{G}\alpha|^2}{n_{\text{ASE}}}]. \tag{44}$$

With the same manner as in the literature [15], the joint distribution of (n_I, n_Q) is given by

$$P_{IQ}(n_I, n_Q) = \int \mathrm{d}^2\eta \mathsf{P}(\eta) K_I(n_I,\eta) K_Q(n_Q,\eta), \tag{45}$$

where

$$\begin{aligned} K_I(n_I,\eta) &= \exp[-\frac{1}{2}(|\eta|^2+\beta^2)] \\ &\quad\times\left|\frac{\eta+\beta}{\eta-\beta}\right|^{n_I} \mathsf{I}_{n_I}[\frac{1}{2}|\eta^2-\beta^2|] \end{aligned} \tag{46}$$

and

$$\begin{aligned} K_Q(n_Q,\eta) &= \exp[-\frac{1}{2}(|\eta|^2+\beta^2)] \\ &\quad\times\left|\frac{\eta+\mathrm{i}\beta}{\eta-\mathrm{i}\beta}\right|^{n_Q} \mathsf{I}_{n_Q}[\frac{1}{2}|\eta^2+\beta^2|]. \end{aligned} \tag{47}$$

Here the kernels (46) and (47) can be obtained from Eqs. (31) and (32). Therefore, we see that the kernels behave according to the *Skellam distribution*. When $\beta \gg |\eta| \gg 1$, the kernels are approximated to as follows:

$$K_I(n_I,\eta) \approx \frac{1}{\sqrt{\pi\beta^2}}\exp[-\frac{|n_I - \mathrm{Re}[\eta]\beta|^2}{\beta^2}] \tag{48}$$

and

$$K_Q(n_Q,\eta) \approx \frac{1}{\sqrt{\pi\beta^2}}\exp[-\frac{|n_Q - \mathrm{Im}[\eta]\beta|^2}{\beta^2}], \tag{49}$$

where we have used Eqs. (35) and (36). Substituting Eqs. (48) and (49) into Eq. (45), we obtain

$$\begin{aligned} &P_{IQ}(n_I, n_Q) \\ \approx\ &\frac{1}{\pi(n_{\text{ASE}}+1)\beta^2} \\ &\times\exp[-\frac{|(n_I+\mathrm{i}n_Q) - \sqrt{G_{\text{net}}}\alpha\beta|^2}{(n_{\text{ASE}}+1)\beta^2}], \end{aligned} \tag{50}$$

when $\beta \gg \sqrt{G_{\text{net}}}|\alpha| \gg 1$.

Like in the previous section, we introduce $z_I = n_I/\beta$ and $z_Q = n_Q/\beta$ to rescale the outcomes by β. This leads us to the following result:

$$\begin{aligned} p_{IQ}(z_I, z_Q) &\approx \frac{1}{\pi(n_{\text{ASE}}+1)} \\ &\quad\times\exp[-\frac{|z-\sqrt{G_{\text{net}}}\alpha|^2}{n_{\text{ASE}}+1}] \\ &= \frac{1}{\pi}\langle z|\hat{\rho}(\alpha)|z\rangle. \end{aligned} \tag{51}$$

This is identical to Eq. (8).

IV. Symbol Error Rate for M-ary PSK

M-ary PSK coherent state signal is defined by

$$|A\exp[\frac{2\pi\mathrm{i}m}{M}]\rangle, \quad m = 0, 1, \cdots, M-1, \tag{52}$$

where $A > 0$. From Eq.(8), the conditional probability density function of measurement outcome $\zeta = \zeta_I + \mathrm{i}\zeta_Q$ for the mth PSK coherent state signal after passing through the channel of N loss-gain optical transmission lines with in-line erbium amplifiers is

$$\begin{aligned} &p(\zeta_I, \zeta_Q|m) \\ &= \frac{1}{2\pi\sigma^2}\exp[-\frac{|\zeta - \sqrt{G_{\text{net}}}A\exp[\mathrm{i}\theta_m]|^2}{2\sigma^2}], \end{aligned} \tag{53}$$

where $\theta_m = 2\pi\mathrm{i}m/M$ and $\sigma^2 = (n_{\text{ASE}}+1)/2$.

From this point, we employ Cahn's method [1] to derive the error probability of the homodyne receiver for M-ary PSK coherent state signal.

Letting $\zeta_I = r\cos\theta$ and $\zeta_Q = r\sin\theta$, Eq. (53) is transformed into

$$\begin{aligned} p(r,\theta|m) &= \frac{r}{2\pi\sigma^2}\exp\Big[-\frac{1}{2\sigma^2}(r^2 + G_{\text{net}}A^2 \\ &\qquad -2r\sqrt{G_{\text{net}}}A\cos[\theta-\theta_m])\Big]. \end{aligned} \tag{54}$$

Integrating with respect to r, we have

$$\begin{aligned} p(\theta|m) &= \int_0^\infty \mathrm{d}r \cdot p(r,\theta|m) \\ &= \frac{1}{2\pi}\exp[-\frac{\gamma}{2}]\left\{1+\sqrt{4\pi}\sqrt{\frac{\gamma}{2}}\cos[\theta-\theta_m] \right. \\ &\quad \times\exp[\frac{\gamma}{2}\cos^2[\theta-\theta_m]] \\ &\quad \left. \times\left(\frac{1}{2}+\frac{1}{2}\mathsf{erf}[\frac{1}{\sqrt{2}}\sqrt{2\frac{\gamma}{2}}\cos[\theta-\theta_m]]\right)\right\}, \end{aligned} \tag{55}$$

where the signal-to-noise ratio (SNR) $\gamma = G_{\mathrm{net}}A^2/\sigma^2$ and the error function $\mathsf{erf}[x] = (2/\sqrt{\pi})\int_0^x \mathrm{d}t \cdot \exp[-t^2]$. Here we let $m = 0$. If γ is sufficiently large, it can be approximated as

$$p(\theta|m=0) \approx \frac{\cos\theta}{\sqrt{\pi(2/\gamma)}}\exp[-\frac{\sin^2\theta}{(2/\gamma)}] \tag{56}$$

for $|\theta| < \pi/2$ (or $\cos\theta > 0$), where the approximation $\mathsf{erf}[x] = 1-\mathsf{erfc}[x] \approx 1-\exp[-x^2]/x\sqrt{\pi}$ for $x \gg 1$ has been used. Therefore, the correct detection probability of the 0th signal is

$$\begin{aligned} P(0|0) &= \int_{-\pi/M}^{\pi/M}\mathrm{d}\theta\cdot p(\theta|0) \\ &\approx \mathsf{erf}[\sqrt{\frac{\gamma}{2}}\sin[\frac{\pi}{M}]], \end{aligned} \tag{57}$$

where $\int \mathrm{d}x \cdot \exp[-ax^2] = (\sqrt{\pi/a}/2)\mathsf{erf}[\sqrt{a}x] + \mathrm{const}$ has been used. Because of the symmetry of the signal constellations, we have $P(m|m) = P(0|0)$ for every m. Therefore, the average probability of correct detection is

$$\begin{aligned} \bar{P}_{\mathrm{c}}(M\text{-ary PSK}) &= \frac{1}{M}\sum_{m=0}^{M-1}P(m|m) = P(0|0) \\ &\approx \mathsf{erf}[\sqrt{\frac{\gamma}{2}}\sin[\frac{\pi}{M}]], \end{aligned} \tag{58}$$

and hence the average probability of error is

$$\begin{aligned} \bar{P}_{\mathrm{e}}(M\text{-ary PSK}) &= 1-\bar{P}_{\mathrm{c}}(M\text{-ary PSK}) \\ &\approx \mathsf{erfc}[\sqrt{\frac{\gamma}{2}}\sin[\frac{\pi}{M}]]. \end{aligned} \tag{59}$$

V. Conclusion

We gave an approximation formula of the error probability of M-ary PSK coherent state signal by a homodyne receiver in the case that the communication channel contains in-line erbium amplifiers, by means of the calculation method in the literature by Cahn [1]. Further, we simply verified the relationship between the POVM $\{(1/\pi)|\zeta\rangle\langle\zeta|\}$ and eight-port homodyne detector in terms of the Skellam distribution.

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Appendix

A. Skellam distribution

In 1937, Irwin considered the distribution of the difference between two independent Poisson random variables with the same parameter [16]. Skellam successfully removed the condition that two Poisson random variables have the same parameter [5]. Now such a distribution is called the *Skellam distribution*. Further, Fisz showed that the Skellam distribution converges to the Gaussian distribution when the parameters are large enough [17]. Here we summarize their results according to the literature [17].

Let X and Y be independent Poisson variables with respective parameters $\lambda_X > 0$ and $\lambda_Y > 0$:

$$P_X(\ell) = \exp[-\lambda_X]\frac{\lambda_X^\ell}{\ell!}, \quad \ell = 0, 1, 2, \dots \tag{60}$$

and

$$P_Y(m) = \exp[-\lambda_Y]\frac{\lambda_Y^m}{m!}, \quad m = 0, 1, 2, \dots. \tag{61}$$

The characteristic function $\mathcal{C}_X(\xi)$ of the distribution P_X is calculated as

$$\begin{aligned} \mathcal{C}_X(\xi) &= \int_{-\infty}^{\infty}\left(\sum_{\ell=0}^{\infty} P_X(\ell)\delta(x-\ell)\right)\mathrm{e}^{\mathrm{i}\xi x}\mathrm{d}x \\ &= \sum_{\ell=0}^{\infty} P_X(\ell)\mathrm{e}^{\mathrm{i}\xi\ell} \\ &= \exp[-\lambda_X]\sum_{\ell=0}^{\infty}\frac{(\lambda_X \mathrm{e}^{\mathrm{i}\xi})^\ell}{\ell!} \\ &= \exp[\lambda_X(\mathrm{e}^{\mathrm{i}\xi}-1)]. \end{aligned} \tag{62}$$

Similarly, the characteristic function $\mathcal{C}_Y(\xi)$ of the distribution P_Y is given as

$$\mathcal{C}_Y(\xi) = \exp[\lambda_Y(\mathrm{e}^{\mathrm{i}\xi}-1)]. \tag{63}$$

Letting $Y' = -Y$, the characteristic function $\mathcal{C}_{Y'}(\xi)$ of the distribution $P_{Y'}$ is immediately obtained as

$$\mathcal{C}_{Y'}(\xi) = \exp[\lambda_Y(\mathrm{e}^{-\mathrm{i}\xi}-1)]. \tag{64}$$

Therefore the characteristic function $\mathcal{C}_Z$ for the difference of two Poisson variables, $Z = X - Y$, is given as

$$\begin{aligned} \mathcal{C}_Z(\xi) &= \exp[\lambda_X(\mathrm{e}^{\mathrm{i}\xi}-1)]\exp[\lambda_Y(\mathrm{e}^{-\mathrm{i}\xi}-1)] \\ &= \exp[-(\lambda_X+\lambda_Y)+\lambda_X\mathrm{e}^{\mathrm{i}\xi}+\lambda_Y\mathrm{e}^{-\mathrm{i}\xi}]. \end{aligned} \tag{65}$$

From this, the first and second moments of the distribution of Z are given as follows:

$$M_1 = -\mathrm{i}\left.\frac{\mathrm{d}\mathcal{C}_Z(\xi)}{\mathrm{d}\xi}\right|_{\xi=0} = \lambda_X - \lambda_Y, \tag{66}$$

and

$$\begin{aligned} M_2 &= (-\mathrm{i})^2\left.\frac{\mathrm{d}^2\mathcal{C}_Z(\xi)}{\mathrm{d}\xi^2}\right|_{\xi=0} \\ &= (\lambda_X+\lambda_Y)+(\lambda_X-\lambda_Y)^2. \end{aligned} \tag{67}$$

Here we let $A = 2\sqrt{\lambda_X\lambda_Y} > 0$ and $B = \sqrt{\lambda_X/\lambda_Y} > 0$, or $\lambda_X = AB/2$ and $\lambda_Y = A/(2B)$. Then the expression (65) can be arranged to

$$\begin{aligned} \mathcal{C}_Z(\xi) &= \mathrm{e}^{-(\lambda_X+\lambda_Y)}\times\exp[\frac{A}{2}\left\{B\mathrm{e}^{\mathrm{i}\xi}+\left(B\mathrm{e}^{\mathrm{i}\xi}\right)^{-1}\right\}] \\ &= \mathrm{e}^{-(\lambda_X+\lambda_Y)}\sum_{n=-\infty}^{\infty}B^n\mathrm{e}^{\mathrm{i}n\xi}\mathsf{I}_n(A), \end{aligned} \tag{68}$$

where $\mathsf{I}_\nu[z] = \sum_{n=0}^{\infty}(n!\Gamma[n+\nu+1])^{-1}(z/2)^{2n+\nu}$ is the modified Bessel function (of the first kind) [7], and where the formula $\exp[(z/2)(t+t^{-1})] = \sum_{n=-\infty}^{\infty}t^n\mathsf{I}_n(z)$ has been used. Therefore the probability density function of $Z = X - Y$ is calculated as

$$\begin{aligned} p_Z(z) &= \frac{1}{2\pi}\int_{-\infty}^{\infty}\mathrm{e}^{-\mathrm{i}\xi z}\mathcal{C}_Z(\xi)\mathrm{d}\xi \\ &= \mathrm{e}^{-(\lambda_X+\lambda_Y)}\sum_{n=-\infty}^{\infty}\frac{B^n\mathsf{I}_n(A)}{2\pi}\int_{-\infty}^{\infty}\mathrm{e}^{-\mathrm{i}(z-n)\xi}\mathrm{d}\xi \\ &= \mathrm{e}^{-(\lambda_X+\lambda_Y)}\sum_{n=-\infty}^{\infty}B^n\mathsf{I}_n(A)\delta(z-n). \end{aligned} \tag{69}$$

Hence the probability mass function of Z is

$$P_Z(n) = \mathrm{e}^{-(\lambda_X+\lambda_Y)}\left(\frac{\lambda_X}{\lambda_Y}\right)^{n/2}\mathsf{I}_n[2\sqrt{\lambda_X\lambda_Y}] \tag{70}$$

for $n = \cdots, -2, -1, 0, 1, 2, \cdots$ [5]. This distribution has the expected value

$$\begin{aligned} \mu_Z &= \mathtt{E}[Z] \\ &= M_1 \\ &= \lambda_X - \lambda_Y \end{aligned} \tag{71}$$

and the variance

$$\begin{aligned} \sigma_Z^2 &= \mathtt{Var}[Z] \\ &= M_2 - (M_1)^2 \\ &= \lambda_X + \lambda_Y. \end{aligned} \tag{72}$$

Here we let $W = (Z - \mu_Z)/\sigma_Z$ for normalization. Namely,

$$W = \kappa_1 Z + \kappa_2 \tag{73}$$

with constants $\kappa_1 = 1/\sigma_Z$ and $\kappa_2 = \mu_Z/\sigma_Z$. Then the characteristic function of W is given by

$$\mathcal{C}_W(\xi) = \mathcal{C}_{\kappa_1 Z+\kappa_2}(\xi) = \mathcal{C}_Z(\kappa_1\xi)\mathrm{e}^{\mathrm{i}\kappa_2\xi}. \tag{74}$$

When λ_1 and λ_2 are large enough,

$$\begin{aligned} \mathcal{C}_W(\xi) &= \exp[-\frac{\xi^2}{2}+\frac{\mu_Z}{\sigma_Z^3}\cdot\frac{(\mathrm{i}\xi)^3}{3!} \\ &\quad +\frac{1}{\sigma_Z^2}\cdot\frac{(\mathrm{i}\xi)^4}{4!}+\frac{\mu_Z}{\sigma_Z^5}\cdot\frac{(\mathrm{i}\xi)^5}{5!}+\cdots] \\ &\approx \exp[-\frac{\xi^2}{2}]. \end{aligned} \tag{75}$$

Therefore, the random variable W obeys the standard normal distribution $\mathcal{N}(0,1)$ when λ_X and λ_Y are large enough. From this, the distribution of the random variable Z is approximated [17] to

$$P_Z(n) \approx \frac{1}{\sqrt{2\pi(\lambda_X+\lambda_Y)}}\exp[-\frac{\{n-(\lambda_X-\lambda_Y)\}^2}{2(\lambda_X+\lambda_Y)}]. \tag{76}$$

Further, the cumulative distribution function is also approximated [17] to

$$F_Z(z \le n) \approx \frac{1}{\sqrt{2\pi\sigma_Z^2}} \int_{-\infty}^{n+1/2} \mathrm{d}z' \exp[-\frac{\{z' - \mu_Z\}^2}{2\sigma_Z^2}]$$
$$= \frac{1}{2}\mathsf{erfc}[\frac{\mu_Z - n - \frac{1}{2}}{\sigma_Z\sqrt{2}}]$$
$$= \mathsf{Q}[\frac{\mu_Z - n - \frac{1}{2}}{\sigma_Z}], \quad (77)$$

where the complementary error function

$$\mathsf{erfc}[x] = \frac{2}{\sqrt{\pi}} \int_x^{\infty} \mathrm{e}^{-\tau^2} \mathrm{d}\tau,$$

and the Q-function (*e.g.*, [18])

$$\mathsf{Q}[x] = \frac{1}{\sqrt{2\pi}} \int_x^{\infty} \exp[-\frac{\tau^2}{2}]\mathrm{d}\tau = \frac{1}{2}\mathsf{erfc}[\frac{x}{\sqrt{2}}].$$

In the literature by Fisz [17], he showed numerical tables of some concrete cases to justify his approximations (which correspond to Eqs. (76) and (77) in this article). For more intuitive understanding, an example of graph of the approximation (76) is shown in Fig. 3, where $\lambda_X = 10$ and $\lambda_Y = 30$. In this example, the expected value is $\mu_Z = 10 - 30 = -20$ and the variance is $\sigma_Z^2 = 10 + 30 = 40$ (*i.e.*, $3\sigma_Z \sim 19$). From Fig. 3, we observe the approximation works well.

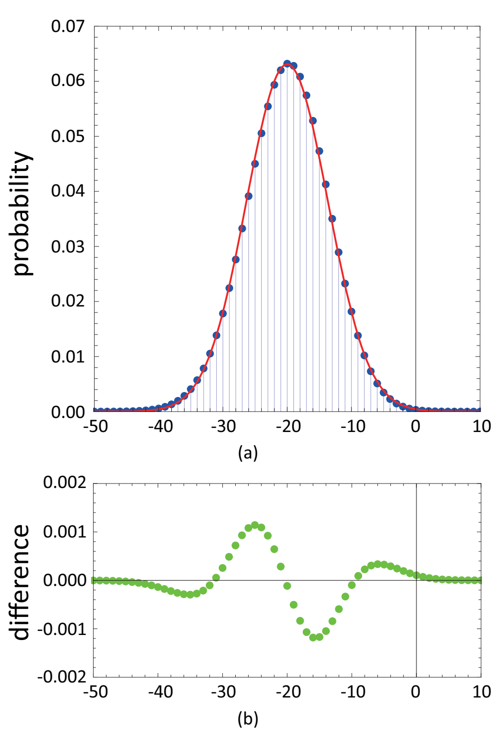

Fig. 3. (color online) Skellam distribution for $\lambda_X = 10$ and $\lambda_Y = 30$. The horizontal axis stands for n. (a) Skellam distribution (blue) and its Gaussian approximation (red). (b) Difference between the true and approximated values (green).