Ideal quantum reading of optical memories

Michele Dall'Arno,[1, 2, 3] Alessandro Bisio,[3, 4] and Giacomo Mauro D'Ariano[3, 4]

[1]*Graduate School of Information Science, Nagoya University, Chikusa-ku, Nagoya, 464-8601, Japan*
[2]*ICFO-Institut de Ciencies Fotoniques, Mediterranean Technology Park, E-08860 Castelldefels (Barcelona), Spain*
[3]*Quit group, Dipartimento di Fisica "A. Volta", via Bassi 6, I-27100 Pavia, Italy*
[4]*Istituto Nazionale di Fisica Nucleare, Gruppo IV, via Bassi 6, I-27100 Pavia, Italy*

Quantum reading is the art of exploiting the quantum properties of light to retrieve classical information stored in an optical memory with low energy and high accuracy. Focusing on the ideal scenario where noise and loss are negligible, we review previous works on the optimal strategies for minimal-error retrieving of information (ambiguous quantum reading) and perfect but probabilistic retrieving of information (unambiguous quantum reading). The optimal strategies largely overcome the optimal coherent protocols (reminiscent of common CD readers), further allowing for perfect discrimination. Experimental proposals for optical implementations of optimal quantum reading are provided.

I. INTRODUCTION

In the engineering of optical memories (such as CDs or DVDs) and readers, a tradeoff among several parameters must be taken into account. High precision in the retrieving of information is surely an indefeasible assumption, but also energy requirements, size and weight can play a very relevant role for applications. Clearly, size and weight of the device increase with the energy, and using a low energetic radiation to read information reduces the heating of the physical bit, thus allowing for smaller implementation of the bit itself. Moreover, many physical media (e.g., superconducting devices) dramatically change their optical properties if the energy flow overcomes a critical threshold.

In the problem of quantum reading [1–9] of optical devices one's task is to exploit the quantum properties of light in order to retrieve some classical digital information stored in the optical properties of a given media, making use of as few energy as possible. The quantum reading of optical memories was first introduced in Ref. [1]. A realistic model of digital memory was considered, where each cell is composed of a beamsplitter with two possible reflectivities. A single optical port is available to probing the beam splitter, while the other port introduces thermal noise in the reading process, so that the problem considered is the discrimination of two lossy and thermal Gaussian channels. It was shown that, for fixed mean number of photons irradiated over each memory cell, even in the presence of noise and loss, a quantum source of light can retrieve more information than any classical source - in particular in the regime of few photons and high reflectivities. This provided the first evidence that the use of quantum light can provide great improvements in applications in the technology of digital memories such as CDs or DVDs.

In practical implementations noise can sometimes be noticeably reduced [10]. On the other hand, in general loss inherently affects quantum optical setups. Nevertheless, a theoretical analysis of the ideal, i.e. lossless and noiseless, quantum reading provides a theoretical insight of the problem and a meaningful benchmark for any experimental realization. In this hypothesis quantum reading of optical devices can be recasted to a discrimination among optical devices with low energy and high precision.

In the ideal reading of a classical bit of information from an optical memory, namely in the discrimination of a quantum optical device from a set of two, different scenarios can be distinguished. A possibility is the on-the-fly retrieving of information (e.g. multimedia streaming), where the requirement is that the reading operation is performed fast - namely, only once, but a modest amount of errors in the retrieved information is tolerable. This scenario corresponds to the problem of minimum energy ambiguous discrimination of optical devices [14–16], where one guesses the unknown device and the task is to minimize the probability of making an error.

On the other hand, in a situation of criticality of errors and very reliable technology, the perfect retrieving of information is an issue. Then, unambiguous discrimination of optical devices [17], where one allows for an inconclusive outcome (while, in case of conclusive outcome, the probability of error is zero) becomes interesting.

In Ref. [2] an optimal strategy for the first scenario - namely, the minimum energy ambiguous discrimination of optical devices - has been provided for the ideal case. This strategy, that exploits fundamental properties of the quantum theory such as entanglement, allows for the ambiguous discrimination of beamsplitters with probability of error under any given threshold, while minimizing the energy requirement. The proposed optimal strategy has been compared with a coherent strategy, reminiscent of the one implemented in common CD readers, showing that the former saves orders of magnitude of energy if compared with the latter, and moreover allows for perfect discrimination with finite energy.

In Ref. [6] the results of Ref. [2] were extended to the case of unambiguous ideal quantum reading - namely, the minimum energy unambiguous discrimination of optical devices. The optimal strategy for unambiguous discrimination of beamsplitters with probability of failure under a given threshold, while minimizing the energy re-

quirement, was provided. It was shown that the optimal strategy does not require any ancillary mode - while in the presence of noise and loss ancillary states improve the performance of the quantum reading setup [1, 18]. Both strategies for ambiguous and unambiguous quantum reading reduce to the same optimal strategy for perfect discrimination if the probability of error (in the former case) or the probability of failure (in the latter case) is set to zero. Then, some experimental setups implementing such optimal strategies which are feasible with present day quantum optical technology, in terms of preparations of single-photon input states, linear optics and photodetectors, were provided.

This paper comprehensively reviews the main results that we obtained in Refs. [2, 6] reformulating them in a coherent and homogeneous presentation. The paper is structured as follows. In Sec. II we formally introduce and discuss the ideal quantum reading of optical memories. In Sec. III we consider the particular case where each memory cell is represented by a beamsplitter. In Sec. IV we compare the optimal quantum reading strategy with the optimal coherent protocol, making use of coherent input states and homodyne detection. In Sec. V we propose some experimental optical implementations of quantum reading. We conclude summarizing our results in Sec. VI.

II. IDEAL QUANTUM READING OF OPTICAL MEMORIES

A M-modes quantum optical device [19] is described by a unitary operator U relating M input optical modes with annihilation operators a_i on $\mathcal{H}_i$, to M output optical modes with annihilation operators a_i' on $\mathcal{H}_{i'}$, where $\mathcal{H}_i$ denotes the Fock space of the optical mode i. We denote the total Fock space as $\mathcal{H} = \bigotimes_i \mathcal{H}_i$.

An optical device is called *linear* if the operators of the output modes are related to the operators of the input modes by a linear transformation, namely

$$\begin{pmatrix} \mathbf{a}' \\ \mathbf{a}'^\dagger \end{pmatrix} = S \begin{pmatrix} \mathbf{a} \\ \mathbf{a}^\dagger \end{pmatrix}, \qquad S := \begin{pmatrix} A & B \\ \bar{B} & \bar{A} \end{pmatrix} \qquad (1)$$

where S is called scattering matrix, $\bar{X}$ denotes the complex conjugate of X, $\mathbf{a} = (a_1, \dots a_N)$ is the vector of annihilation operators of the input mode, and analogously $\mathbf{a}'$ for the output modes. If $B = 0$ in Eq. (1) the device is called *passive* and conserves the total number of photons, that is $\langle \psi | N | \psi \rangle = \langle \psi | U^\dagger N U | \psi \rangle$ with $N := \sum_i a_i^\dagger a_i$ the *number operator* on $\mathcal{H}$. In the following, for any pure state $|\psi\rangle$, we denote with $\psi := |\psi\rangle \langle \psi|$ the corresponding projector. For any Fock space $\mathcal{H}$, we denote with $|n\rangle$ a Fock basis in $\mathcal{H}$ ($|0\rangle$ denotes the state of the vacuum).

Suppose we want to discriminate between two linear optical passive devices U_1 and U_2. If a single use of the unknown device is available, the most general strategy consists of preparing a bipartite input state $\rho \in \mathcal{B}(\mathcal{H} \otimes \mathcal{K})$

($\mathcal{K}$ is an ancillary Fock space with mode operators b_i), applying locally the unknown device and performing a bipartite POVM Π on the output state $(\mathcal{U}_x \otimes \mathcal{I}_\mathcal{K})\rho = (U_x \otimes I_\mathcal{K})\rho(U_x^\dagger \otimes I_\mathcal{K})$ (x can be either 1 or 2).

$$\begin{array}{c} \mathcal{H} \\ \rho \boxed{\begin{array}{c} \boxed{U_x} \\ \mathcal{K} \end{array}} \boxed{\Pi} \end{array}. \qquad (2)$$

The choice of Π in Eq. (2) depends on the figure of merit taken into account. For example, for ambiguous discrimination $\Pi = \{\Pi_1, \Pi_2\}$ and one's task is to minimize the probability of error

$$P_E(\rho, U_1, U_2) := \mathrm{Tr}[(\mathcal{U}_1 \otimes \mathcal{I}_\mathcal{H})(\rho)\Pi_2 + (\mathcal{U}_2 \otimes \mathcal{I}_\mathcal{H})(\rho)\Pi_1],$$

with $0 \le P_E(\rho, U_1, U_2) \le 1/2$. When $p_1 = p_2 = 1/2$ the minimal probability of error has been proven to be given by the following function [20] of ρ,

$$P_E(\rho^*, U_1, U_2) = \frac{1}{2}\left(1 - ||\left[(\mathcal{U}_1 - \mathcal{U}_2) \otimes \mathcal{I}_\mathcal{K}\right]\rho||_1\right), \quad (3)$$

where $||X||_1 = \mathrm{Tr}[\sqrt{X^\dagger X}]$ denotes the trace norm.

For unambiguous discrimination $\Pi = \{\Pi_1, \Pi_2, \Pi_F\}$, $\mathrm{Tr}[(\mathcal{U}_1 \otimes \mathcal{I}_\mathcal{H})(\rho)\Pi_2] = \mathrm{Tr}[(\mathcal{U}_2 \otimes \mathcal{I}_\mathcal{H})(\rho)\Pi_1] = 0$ and one's task is to minimize the probability of inconclusive outcome (failure)

$$P_F(\rho, U_1, U_2) := \mathrm{Tr}[(\mathcal{U}_1 \otimes \mathcal{I}_\mathcal{H} + \mathcal{U}_2 \otimes \mathcal{I}_\mathcal{H})(\rho)\Pi_F], \quad (4)$$

with $0 \le P_F(\rho, U_1, U_2) \le 1$.

In the following, whenever the results we present hold for $P_E(\rho, U_1, U_2)$ (in an ambiguous discrimination scenario) as well as for $P_F(\rho, U_1, U_2)$ (in an unambiguous discrimination scenario), we will simply write $P(\rho, U_1, U_2)$.

Upon denoting with $E_D(\rho) := \mathrm{Tr}[\rho(N \otimes I_\mathcal{K})]$ the energy that flows through the unknown device, the total energy of the input state is $E(\rho) := E_D + \mathrm{Tr}[\rho(I_\mathcal{H} \otimes N_\mathcal{K})]$.

We can now introduce the ideal quantum reading problem [2, 6]. For any set of two optical devices $\{U_1, U_2\}$ and any threshold q in the probability of error (failure), find the minimum energy input state ρ^* that allows to ambiguously (unambiguously) discriminate between U_1 and U_2 with probability of error (failure) not greater than q, namely

$$\rho^* = \arg \min_{\rho \text{ s.t. } P(\rho, U_1, U_2) \le q} E(\rho). \qquad (5)$$

where $P(\rho, U_1, U_2) = P_E(\rho, U_1, U_2)$ for the ambiguous discrimination problem and $P(\rho, U_1, U_2) = P_F(\rho, U_1, U_2)$ for the unambiguous discrimination problem.

First, notice that for any POVM Π we have $P((\mathcal{U}_1 \otimes \mathcal{I}_\mathcal{K})\rho, I, U_2 U_1^\dagger) = P(\rho, U_1, U_2)$ and $E((\mathcal{U}_1 \otimes \mathcal{I}_\mathcal{K})\rho) = E(\rho)$, so we can restrict our analysis to the case in which $U_1 = I$ and $U_2 = U$, and identify $P(\rho, I, U) = P(\rho, U)$.

Then, notice that without loss of generality the constraint in Eq. (5) can be restated as $P(\rho, U) = q$. Indeed,

for any POVM Π we have that $P(\rho, U)$ is a continuous function maximized in $|0\rangle \langle 0|$ [indeed $P_E(|0\rangle \langle 0|, U) = 1/2$ and $P_F(|0\rangle \langle 0|, U) = 1$]. So for any ρ with $P(\rho, U) < q$ there exists a $0 < \alpha \leq 1$ such that $P[(1 - \alpha)\rho + \alpha |0\rangle \langle 0|, U] = q$. Since $E[(1 - \alpha)\rho + \alpha |0\rangle \langle 0|] < E(\rho)$, the constraint in Eq. (5) becomes $P(\rho, U) = q$.

Proposition 1 (Optimal state is pure). *For any optical device U and any threshold q in the probability of error $P_E(\rho, U)$ [probability of failure $P_F(\rho, U)$], there exists a state ρ^* which minimizes Eq. (5) such that ρ^* is pure.*

Proof. Notice that Eq. (5) is equivalent to $C(\rho, U) := pP(\rho, U) + (1 - p)E(\rho)$, for any fixed value of p. If ρ^* is the state that minimizes $C(\rho, U)$, for $q := P(\rho^*, U)$ we have that $E(\rho^*)$ gives the minimum possible value for the energy. Since $P(\rho, U)$ and $E(\rho)$ are linear functions of ρ, it follows that $C(\rho, U)$ is a linear function of ρ and its minimum is attained on the boundary of its domain, namely for a pure state $|\psi^*\rangle$. $\square$

As a consequence of Proposition 1, Eq. (5) can be restated as

$$\psi^* = \arg \min_{\psi \text{ s.t. } P(\psi, U) = q} E(\psi). \qquad (6)$$

For pure states, the probability of error in the ambiguous discrimination when $p_1 = p_2 = 1/2$ given by Eq. (3) becomes

$$P_E = \frac{1}{2}\left(1 - \sqrt{1 - |\langle \psi| (U \otimes I_\mathcal{K}) |\psi\rangle|^2}\right). \qquad (7)$$

For pure states, the probability of failure in the unambiguous discrimination when $p_1 = p_2 = 1/2$ given by Eq. (4) has been proved to be given by [17]

$$P_F(\psi^*, U) = |\langle \psi| U \otimes I_\mathcal{K} |\psi\rangle|. \qquad (8)$$

Proposition 2 (No ancillary modes are required). *For any optical device U and any threshold q in the probability of error $P_E(\rho, U)$ [probability of failure $P_F(\rho, U)$], there exists a state ρ^* which minimizes Eq. (5) such that $\rho^* \in \mathcal{H}$.*

Proof. We show that for any pure input state ψ there exists a pure state ψ' that does not resort to ancillary modes and that allows for quantum reading with the same probability of error (failure) but with lower energy. Let us denote with $|\mathbf{n}\rangle = |n_1, \ldots, n_{\dim \mathcal{H}}\rangle$ a Fock basis in $\mathcal{H}$ with respect to which U is diagonal and with $|\mathbf{m}\rangle = |m_1, \ldots, m_{\dim \mathcal{K}}\rangle$ a Fock basis in $\mathcal{K}$. Let us denote with $e^{i\delta_i}$ the eigenvalue of U corresponding to mode i-th, and let $\boldsymbol{\delta} = (\delta_1, \ldots, \delta_{\dim \mathcal{H}})$, namely $U |\mathbf{n}\rangle = e^{i\boldsymbol{\delta} \cdot \mathbf{n}} |\mathbf{n}\rangle$. Any pure input state can be written as $|\psi\rangle = \sum_{\mathbf{n}, \mathbf{m}} c_{\mathbf{n}, \mathbf{m}} |\mathbf{n}, \mathbf{m}\rangle$ for some $c_{\mathbf{n}, \mathbf{m}}$, then one has $\langle \psi| U \otimes I_\mathcal{K} |\psi\rangle = \sum_{\mathbf{n}, \mathbf{m}} |c_{\mathbf{n}, \mathbf{m}}|^2 e^{i\boldsymbol{\delta} \cdot \mathbf{n}}$ and $E(\psi) = \sum_{\mathbf{n}, \mathbf{m}} |c_{\mathbf{n}, \mathbf{m}}|^2 (\sum_i n_i + \sum_j m_j)$. For any $|\psi\rangle$ let us define $|\psi'\rangle := \sum_{\mathbf{n}} c'_{\mathbf{n}} |\mathbf{n}, \mathbf{0}\rangle$ with $c'_{\mathbf{n}} := \sqrt{\sum_{\mathbf{m}} |c_{\mathbf{n}, \mathbf{m}}|^2}$, then one has $\langle \psi'| U \otimes I_\mathcal{K} |\psi'\rangle = \langle \psi| U \otimes I_\mathcal{K} |\psi\rangle$ and $E(\psi') =$

$\sum_{\mathbf{n}, \mathbf{m}} |c_{\mathbf{n}, \mathbf{m}}|^2 (\sum_i n_i)$. Since $P(\psi', U) = P(\psi, U)$ and $E(\psi') \leq E(\psi)$ - the former immediately following from Eq.s (7) and (8) - the statement is proved. $\square$

Since no ancillary modes are required, the energy $E_D(\psi)$ that flows through the unknown device is equal to the total energy of the input state $E(\rho)$, so minimizing the former instead than the latter - namely, replacing $E(\psi)$ with $E_D(\psi)$ in Eq. (6) - does not change the optimal state.

III. IDEAL QUANTUM READING OF BEAMSPLITTERS

A beamsplitter is a two-mode linear passive quantum optical device such that $A \in SU(2)$ in Eq. (1). In the following we will use the basis $\{|n, m\rangle\}$ with respect to which A is diagonal with eigenvalues $e^{\pm i\delta}$, $0 \leq \delta \leq \pi$. With this choice, for any $|\psi\rangle = \sum_{n,m=0}^{\infty} \alpha_{n,m} |n, m\rangle$, we have $U |n, m\rangle = e^{i\delta(n-m)} |n, m\rangle$, so that $\langle \psi| U |\psi\rangle = \sum_{n,m=0}^{\infty} |\alpha_{n,m}|^2 e^{i\delta(n-m)}$ and $\langle \psi| N |\psi\rangle = \sum_{n,m=0}^{\infty} |\alpha_{n,m}|^2 (n+m)$. We notice that both these expressions only depend on the squared modulus of the coefficients $\alpha_{n,m}$, so we can assume $\alpha_{n,m}$ to be real and positive.

Here $\lfloor x \rfloor$ ($\lceil x \rceil$) denotes the maximum (minimum) integer number smaller (greater) than x.

Proposition 3 (Optimal quantum reading of beamsplitters). *For any beamsplitter U and for any threshold q in the probability of error (probability of failure), there exists a state ψ^* which minimizes Eq. (6) such that*

$$|\psi^*\rangle = \alpha \frac{|0, n^*\rangle + |n^*, 0\rangle}{\sqrt{2}} + \sqrt{1 - \alpha^2} |00\rangle, \qquad (9)$$

where

$$|\alpha| = \sqrt{\frac{1 - K(q)}{1 - \cos(\delta n^*)}},$$

$$K(q) = \begin{cases} 2\sqrt{q(1 - q)} & \text{for ambiguous reading} \\ q & \text{for unambiguous reading} \end{cases},$$

$$n^* = \arg \min_{\lfloor x^* \rfloor, \lceil x^* \rceil} E(\psi^*),$$

$$x^* = \min(x > 0 | \delta x = \tan(\delta x/2)).$$

Proof. First we prove that the optimal state in Eq. (6) is a superposition of NOON states. For any state $|\psi\rangle = \sum_{n,m} \alpha_{n,m} |n, m\rangle$, the state $|\psi'\rangle = \sqrt{1/2} \sum_l \alpha'_l (|l, 0\rangle + |0, l\rangle)$ with $|\alpha'_l|^2 = \sum_{|n-m|=l} |\alpha_{nm}|^2$ is such that

$$\langle \psi'| N |\psi'\rangle = \sum_{n,m=0}^{\infty} \alpha_{nm}^2 |n - m| \leq \langle \psi| N |\psi\rangle, \qquad (10)$$

$$|\langle \psi'| U |\psi'\rangle| = \left| \sum_{n,m=0}^{\infty} \alpha_{nm}^2 \cos(\delta|n - m|) \right| \leq |\langle \psi| U |\psi\rangle|.$$

So we have $\langle\psi|\,U\,|\psi\rangle \in \mathbb{R}$ and the constraint in Eq. (6) becomes $\langle\psi|\,U\,|\psi\rangle = K(q)$.

Then we prove that the optimal state is the superposition of two NOON states. Let $|\psi^*\rangle = \sqrt{1/2}\sum_n \alpha_n^*(|n,0\rangle + |0,n\rangle)$ be the optimal state and let the set $\{\alpha_n^*\}$ have $N \geq 3$ not-null elements. Then there exist n_1 and n_2 such that $\alpha_{n_1}, \alpha_{n_2} \neq 0$ and $\cos(\delta n_1) \leq K(q) \leq \cos(\delta n_2)$. Define $|\chi\rangle := 1/\sqrt{2}\sum_{i=1,2}\beta_{n_i}(|n_i,0\rangle + |0,n_i\rangle)$ such that $\langle\chi|\,U\,|\chi\rangle = K(q)$, and $|\xi\rangle := 1/\sqrt{2}(1-\epsilon)^{-1/2}\sum_n \gamma_n(|n,0\rangle + |0,n\rangle)$, where

$$\gamma_n = \begin{cases} \alpha_n & \text{if } n \neq n_1, n_2 \\ \sqrt{\alpha_n^2 - \epsilon\beta_n^2} & \text{if } n = n_1, n_2 \end{cases},$$

and $\epsilon \leq \min(\alpha_{n_1}/\beta_{n_1}, \alpha_{n_2}/\beta_{n_2})$. Notice that $\langle\xi|\,U\,|\xi\rangle = K(q)$, and $\langle\psi^*|\,N\,|\psi^*\rangle = \epsilon\langle\chi|\,N\,|\chi\rangle + (1-\epsilon)\langle\xi|\,N\,|\xi\rangle$. If $\langle\chi|\,N\,|\chi\rangle = \langle\psi^*|\,N\,|\psi^*\rangle$ the statement follows with $|\psi\rangle = |\chi\rangle$. If $\langle\chi|\,N\,|\chi\rangle \neq \langle\psi^*|\,N\,|\psi^*\rangle$, either $\langle\chi|\,N\,|\chi\rangle < \langle\psi^*|\,N\,|\psi^*\rangle$ or $\langle\xi|\,N\,|\xi\rangle < \langle\psi^*|\,N\,|\psi^*\rangle$, that contradicts the hypothesis that $|\psi^*\rangle$ is the optimal state.

Finally we prove that the optimal state is the superposition of a NOON state and the vacuum. Let $|\psi^*\rangle = 1/\sqrt{2}\sum_{i=1,2}\alpha_{n_i}(|n_i,0\rangle + |0,n_i\rangle)$. Then

$$\langle\psi^*|\,N\,|\psi^*\rangle = \frac{n_2\cos(\delta n_1) - n_1\cos(\delta n_2) + K(q)(n_1 - n_2)}{\cos(\delta n_1) - \cos(\delta n_2)}.$$

It is lengthy but not difficult to verify (see Ref. [2] for an explicit proof) that it is not restrictive to set $n_2 = 0$, so one has $\langle\psi^*|\,N\,|\psi^*\rangle = [1 - K(q)][1 - \cos(\delta n_1)]^{-1}n_1$. Then one can see that it is not restrictive to choose $\pi/2 \leq \delta n_1 \leq \pi$, where $\langle\psi^*|\,N\,|\psi^*\rangle$ can be proven [21] to be a convex function that attains its minimum for $n_1 = \lfloor x^* \rfloor, \lceil x^* \rceil$, with $x^* = \min(x \in \mathbb{R}^+ | \delta x = \tan(\delta x/2))$. The statement immediately follows. $\quad\square$

Notice that from Proposition 3 it immediately follows that ambiguous (unambiguous) discrimination between beamsplitters U and I can be achieved only if the threshold q in the probability of error (failure) satisfies the inequality $K(q) \geq \cos(\delta n^*)$ with $K(q)$ as in the statement of Proposition 3.

The optimal energy-error tradeoffs in the ambiguous and unambiguous quantum reading are trivial consequences of Proposition 3, given respectively by

$$E(P_E) = \frac{1 - 2\sqrt{P_E(1 - P_E)}}{1 - \cos(\delta n^*)}n^* \quad (11)$$

$$E(P_F) = \frac{1 - P_F}{1 - \cos(\delta n^*)}n^*, \quad (12)$$

where n^* is constant (for any fixed δ) and is given in the statement of Proposition 3. Figure 1 shows the optimal energy-error tradeoff for some values of δ.
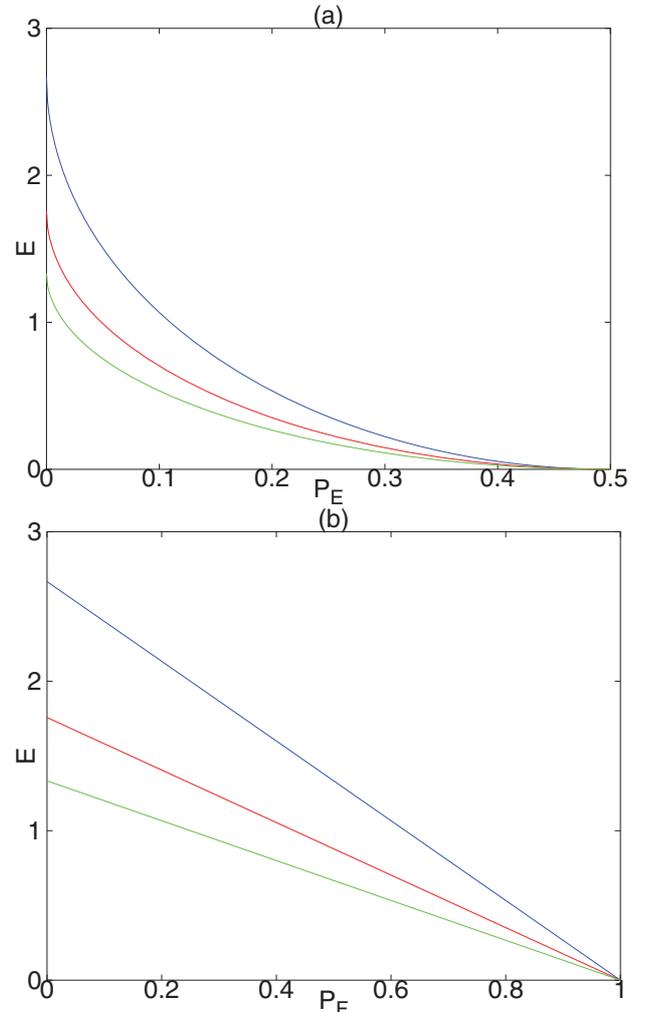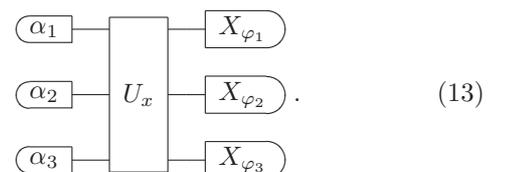

FIG. 1. (Color online) Optimal tradeoff between the energy E and the probability of error P_E (probability of failure P_F) in the ambiguous (a) and unambiguous (b) discrimination of I and $U = \exp(i(\delta a_1^\dagger a_1 - \delta a_2^\dagger a_2))$, for $\delta = \pi/6$ (upper line), $\delta = \pi/4$ (middle line), and $\delta = \pi/3$ (lower line), as given by Eq. (11).

IV. COMPARISON WITH COHERENT STRATEGY

Here we consider the minimum energy discrimination that makes use of coherent input states $|\alpha_i\rangle$ and homodyne detections X_{φ_i} to ambiguously [22] discriminate a single use of a n-modes passive linear optical device randomly chosen in the set $\{I, U\}$ with equal probabilities

$$\qquad\qquad (13)$$

If we consider coherent input states $|\alpha_i\rangle$ on mode i the global input state is $|\xi\rangle = \bigotimes_i |\alpha_i\rangle$ which corresponds to an energy value

$$E(\xi) := \langle\xi| N |\xi\rangle = \sum_i |\alpha_i|^2. \qquad (14)$$

Since for any passive linear device V we have that $V \bigotimes_i |\alpha_i\rangle = \bigotimes_i |\beta_i\rangle$ where $|\beta_i\rangle$ are coherent states, we can assume U to be diagonal, i.e. $U = \sum_i e^{i\delta_i a_i^\dagger a_i}$. The evolution of $|\xi\rangle$ under the action of U is then given by

$$U |\xi\rangle = \bigotimes_i |e^{i\delta_i}\alpha_i\rangle. \qquad (15)$$

A quantum homodyne detection X_φ is described [23–25] by the POVM $\{|x,\varphi\rangle \langle x,\varphi|\}$, where $|x,\varphi\rangle$ are the eigenstates of the quadrature $e^{i\varphi}a + e^{-i\varphi}a^\dagger$. The probability of outcome x when the system is prepared in a coherent state $|\alpha\rangle$ with $\alpha = e^{i\phi_\alpha}|\alpha|$ is given by the Gaussian

$$p_\varphi(x|\alpha) = |\langle\alpha|x,\varphi\rangle|^2 = \sqrt{\frac{2}{\pi}} e^{-2(x-|\alpha|\cos(\varphi+\phi_\alpha))^2}. \quad (16)$$

We notice that $p_\varphi(x|\alpha)$ depends on the phases φ and ϕ_α only through the sum $\varphi + \phi_\alpha$. We can then fix $\varphi = 0$ and vary only the α_i. The conditional probabilities of outcome $\mathbf{x} = (x_i)$ given I or U are n-dimensional Gaussians, namely

$$p(\mathbf{x}|I) = (2/\pi)^{n/2}e^{2|\mathbf{x}-\mathbf{v_0}|^2}, \quad p(\mathbf{x}|U) = (2/\pi)^{n/2}e^{2|\mathbf{x}-\mathbf{v_1}|^2}, \quad (17)$$

with $\mathbf{v_0} = (\operatorname{Re}\alpha_i)$ and $\mathbf{v_1} = (\operatorname{Re}e^{i\delta_i}\alpha_i)$.

Any classical postprocessing of the outcome $\mathbf{x}$ can be described by a function $q(X|\mathbf{x})$ that evaluates to 1 if one guesses the unitary X from outcome $\mathbf{x}$, and to 0 otherwise, with $X = I, U$. The probability of error is given by

$$P_E(\xi) = \frac{1}{2}\int d\mathbf{x}\, p(\mathbf{x}|I)q(U|\mathbf{x}) + p(\mathbf{x}|U)q(I|\mathbf{x}), \quad (18)$$

and thus the optimal postprocessing is

$$q(X|\mathbf{x}) = \begin{cases} 1 & \text{if } p(\mathbf{x}|X) \geq p(\mathbf{x}|Y) \\ 0 & \text{if } p(\mathbf{x}|X) < p(\mathbf{x}|Y) \end{cases}. \quad (19)$$

Inserting Eq. (19) and Eq. (17) into the expression in Eq. (18), the probability of error becomes

$$P_E(\xi) = \frac{1}{2}\left[1 + (2/\pi)^{n/2}\int_A d\mathbf{x}\left(e^{-2|\mathbf{x}-\mathbf{v_0}|^2} - e^{-2|\mathbf{x}-\mathbf{v_1}|^2}\right)\right], \quad (20)$$

where we defined the set

$$A = \{\mathbf{x} \text{ s.t. } |\mathbf{x}-\mathbf{v_0}|^2 \geq |\mathbf{x}-\mathbf{v_1}|^2\}. \quad (21)$$

Within this framework it is more convenient to fix the amount of energy, that is the average number of photons η, and find the input state $|\xi^*\rangle$ that minimizes the probability of error in the discrimination, i.e.

$$|\xi^*\rangle = \arg \min_{\langle\xi|N|\xi\rangle=\eta} P_E(\xi). \qquad (22)$$
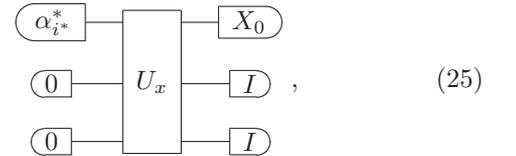
With a little machinery it is possible to prove that $P_E(\xi)$ is a non-increasing function of $|\mathbf{v_0} - \mathbf{v_1}|^2$ and then the minimization of $P_E(\xi)$ can be rephrased as a maximization of $|\mathbf{v_0} - \mathbf{v_1}|^2$. We have then

$$|\mathbf{v_0} - \mathbf{v_1}|^2 = \sum_i [\operatorname{Re}(\alpha_i) - \operatorname{Re}(e^{i\delta}\alpha_i)]^2$$
$$\leq \sum_i [2\sin(\delta_i/2)|\alpha_i|]^2$$
$$\leq 4\sin^2(\delta/2)\eta, \qquad (23)$$

where $\delta^* := \arg\max_{\delta_i}|\delta_i|$, and i^* labels the corresponding mode. The bounds in Eq. (23) are achieved for

$$|\xi^*\rangle = \bigotimes_{i\neq i^*} |0_i\rangle \otimes |\alpha_{i^*}^*\rangle, \qquad (24)$$

where $\alpha_{i^*}^* = \sqrt{\eta}\exp(i\frac{\pi-\delta^*}{2})$. The corresponding optimal discrimination strategy is

$$(25)$$

where $-\boxed{I}$ means that the corresponding mode is discarded. With this choice of the input state the probability of error becomes

$$P_E = \sqrt{\frac{2}{\pi}} \int_{-\infty}^0 dx\, e^{-2\left(x-\sqrt{\eta}\sin\frac{|\delta^*|}{2}\right)^2}$$
$$= \frac{1}{2}\left[1 - \operatorname{erf}\left(\sqrt{2\eta}\sin\frac{|\delta^*|}{2}\right)\right]. \quad (26)$$

where $\operatorname{erf}(x) := \frac{2}{\sqrt{\pi}}\int_0^x dt \exp(-t^2)$ denotes the error function.

From Eq. (26) one can obtain the tradeoff between the energy and the probability of error, which is plotted in Fig. 2, for some choices of U_1 and U_2. If we consider the case in which we want to discriminate a 50/50 beamsplitter from the identity, one can notice that, for $P_E = 0.1$, the coherent state - homodyne detection discrimination strategy requires a factor of ~ 4 more photons that the optimal strategy. Moreover, this factor increases as the two devices get closer, i. e. for small values of δ. For example, when $\delta = \pi/12$, the factor is ~ 12. As expected, one notice that this factor increases when the probability of error decreases.
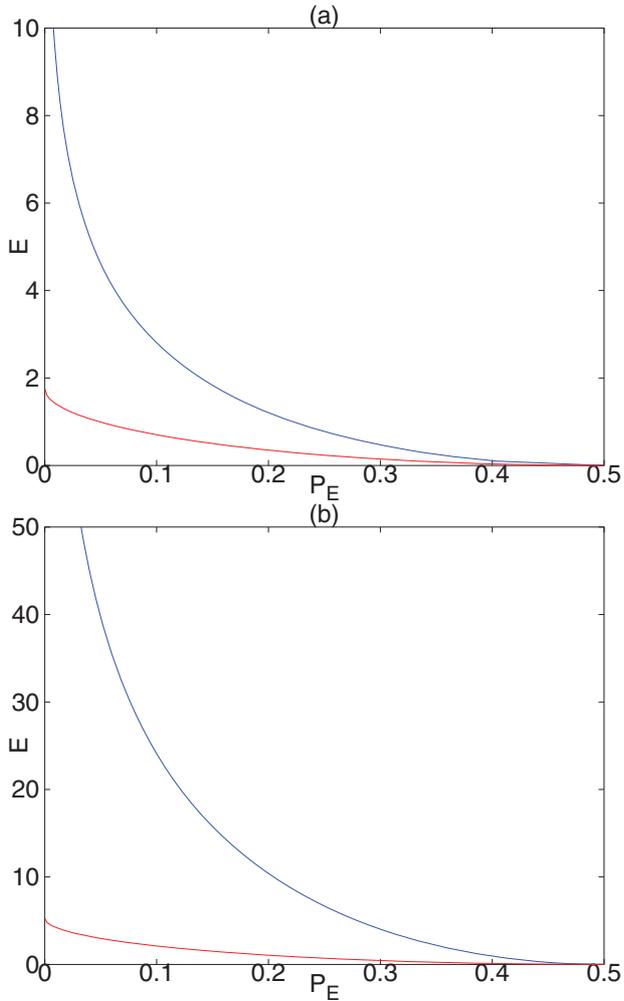
FIG. 2. (Color online) Optimal tradeoff between the energy E and the probability of error P_E in the discrimination of I and $U = \exp(i(\delta a_1^\dagger a_1 - \delta a_2^\dagger a_2))$ ($\delta = \pi/4$ in (a) and $\delta = \pi/12$ in (b)). The upper line represents the discrimination with coherent states and homodyne detections, while the lower line represents the optimal discrimination. Comparing (a) and (b), we notice that the improvement provided by the optimal strategy increases as δ decreases.

V. EXPERIMENTAL SETUP FOR QUANTUM READING

In this Section we provide experimental setups for ambiguous, unambiguous, and perfect quantum reading, which are feasible with present quantum optical technology. The input is a single-photon state, that can be realized e.g. through spontaneous parametric down conversion or through the attenuation of a laser beam. The evolution is given by a circuit of beamsplitters, one of which is the unknown one, and the final measurement is implemented through photodetectors.

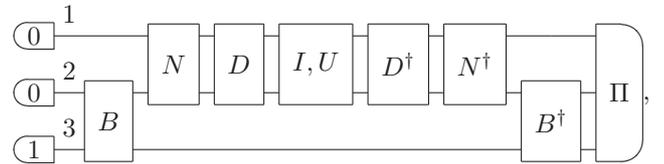In Proposition 2 we proved that, for the ambiguous (unambiguous) quantum reading of optical devices, no ancillary modes are required. Nevertheless, the proposed setups for quantum reading make use of three-modes input states - namely, an ancillary mode is employed. This choice is due to the requirement to have an input state with fixed number of photons in order to be able to experimentally take into account loss. For this reason, our setup minimizes the energy $E_D(\rho)$ that flows through the unknown device, while the total energy of the input state is fixed.

In the following, for any beamsplitter X we denote with A_X the A matrix of X in Eq. (1), so we write

$$A_X = \begin{pmatrix} r_X & -t_X \\ t_X & r_X \end{pmatrix}, \qquad A_X^\dagger = \begin{pmatrix} r_X & t_X \\ -t_X & r_X \end{pmatrix}.$$

We define the reflectivity R_X and the transmittivity T_X of X as $R_X := |r_X|^2$ and $T_X := |t_X|^2$, respectively, with $R_X + T_X = 1$.

The general setup is given by a Mach-Zender interferometer with beamsplitters B and $B^\dagger$, acting on modes 2 and 3. In one of the harms of the interferometer (corresponding to mode 2), the following beamsplitters are inserted
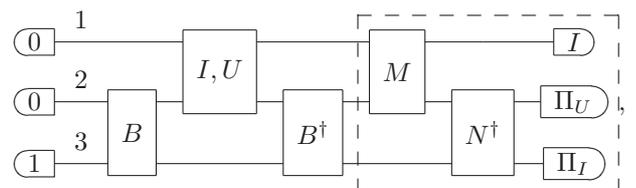

where N is a 50/50 beamsplitter, I, U is the unknown beamsplitter, and D is the beamsplitter diagonalizing U. The POVM Π is different for ambiguous and unambiguous quantum reading. It is easy to verify that the composition of beamsplitters DN reduces to a phase shifter on mode 2, namely

$$A_D = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 1 \\ i & -i \end{pmatrix}, \qquad A_D A_N = \begin{pmatrix} 1 & 0 \\ 0 & i \end{pmatrix}. \quad (27)$$

It is easy to check that this phase shifter is irrelevant, so in the following we will disregard it.

Here we describe an experimental setup implementing the optimal strategy for ambiguous quantum reading as given by Proposition 3, namely the ambiguous discrimination of a beamsplitter randomly chosen from the set $\{I, U\}$ with equal prior probabilities, with probability of error $P_E(\rho, U)$ under a given threshold q and minimal energy flow through the unknown device. In the following we set $K(q) := 2\sqrt{q(1-q)}$. According to Proposition 3, in order to have $P_E(\rho, U) \leq q$, we must have $K(q) \geq \sqrt{R_U}$.

The experimental setup is then given by

where the reflectivities and transmittivities of beamsplitters B, M and $N^\dagger$ are given by

$$R_B = \frac{K(q) - r_U}{1 - r_U},$$
$$R_M = \frac{[1 - K(q)][K(q) - r_U]}{(1 - 2q)^2},$$
$$R_N = \sqrt{1 - q}.$$
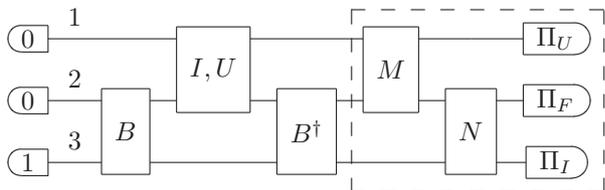
The optimal measurement for ambiguous discrimination [20] is implemented by the two beamsplitters M and $N^\dagger$ and by the two photocounters Π_U and Π_I surrounded by the dashed line (no measurement is performed on output mode 1). The conditional probabilities $p_{X|Y}$ of detecting a photon in photodetector Π_X given that the unknown device is Y are given by

$$p_{U|U} = p_{I|I} = 1 - q, \qquad p_{I|U} = p_{U|I} = q.$$

Detecting a photon in Π_U or Π_I implies that the unknown beamsplitter is U or I, respectively, with probability of error q.

Here we describe an experimental setup implementing the optimal strategy for unambiguous quantum reading as given by Proposition 3, namely the unambiguous discrimination of a beamsplitter randomly chosen from the set $\{I, U\}$ with equal prior probabilities, with probability of failure $P_F(\rho, U)$ under a given threshold q and minimal energy flow through the unknown device. According to Proposition 3, in order to have $P_F(\rho, U) \leq q$, we must have $q \geq \sqrt{R_U}$.

The experimental setup is given by

where the reflectivities and transmittivities of beamsplitters B, M and N are given by

$$R_B = \frac{q - r_U}{1 - r_U},$$
$$R_M = \frac{[\sqrt{1 + r_U} - \sqrt{q(q - r_U)}]^2}{(1 + q)^2},$$
$$R_N = \sqrt{1 - q}.$$

The optimal measurement for unambiguous discrimination [17] is implemented by the two beamsplitters M and N and by the three photocounters Π_U, Π_I, and Π_F surrounded by the dashed line. The conditional probabilities $p_{X|Y}$ of detecting a photon in photodetector Π_X given that the unknown device is Y are given by

$$p_{U|U} = p_{I|I} = 1 - q,$$
$$p_{I|U} = p_{U|I} = 0,$$
$$p_{F|U} = p_{F|I} = q.$$

Detecting a photon in Π_U or Π_I implies that the unknown beamsplitter is certainly U or I, respectively, while detecting a photon in Π_F declares a failure with probability q.

VI. CONCLUSION

In this paper we considered ambiguous and unambiguous quantum reading of optical memories, on the assumption that noise and loss are negligible (Sec. II). We provided the optimal strategy for ambiguous and unambiguous quantum reading of beamsplitters (Sec. III), showing that the optimal input state is a superposition of a NOON state and the vacuum. In Sec. IV we showed that the optimal strategy for ambiguous quantum reading largely overcomes the optimal coherent protocol (reminiscent of common CD readers), further allowing for perfect quantum reading. Finally in Sec. V we proposed some experimental implementations of ambiguous and unambiguous quantum reading, where the input state was fixed to be a single photon state. By making use of an ancillary mode it was possible to tune the amount of energy flowing through the device.

In addition to their relevance in the framework of quantum communication and information theory, the presented results also have obvious connections with experimental quantum optical applications. For these reasons we believe that they will have a relevant impact in the future development of technology for storage and retrieval of digital information.

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