The enhanced phase estimation using quantum Fisher information in nonclassical continuous-variable states and its application

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We present a generalised form of optimal phase estimation using quantum Fisher information in two-mode continuous-variable states with linear and nonlinear phase shifts. Two important examples of continuous-variable states are rigorously investigated for optimal phase estimation. Toward the practical application of the states, the idea of calibrating the strength of nonlinearity has been proposed in a nonliear medium.

I. INTRODUCTION

Since the beginning of development of quantum optics, it has been recognised that quantum metrology is an important field not only for examining the fundamental characteristics of measurements in quantum physics but also for practical applications using entanglement [1–4]. Many interesting properties in nonclassical entangled states have been examined in the field of quantum optics and quantum information [5], and particularly, the nonclassical two-mode states are of use for quantum phase enhancement in optical interferometry [3, 4].

For discrete nonclassical quantum states, a two-mode path-entangled Fock state (called NOON state) has clearly shown that the quantumness can beat a classical limit of optimal phase estimation [12]. Due to the difficulties of implementing a large photon NOON state and of maintaining its robustness against particle losses [13], other nonclassical discrete quantum states have studied such as BAT states, which are built by passing a number-squeezed state through a beam splitter (BS) [14], and mm' states consisting m photons in mode 1 and m' in mode 2 and vice versa [15].

Several continuous-variable (CV) quantum states have been also investigated for quantum metrology [1, 16]. In addition to the advantages of their practical feasibility, one of the primary advantages is that we can explore the parameters space of the quantum states continuously in contrast to discrete one. Since the squeezed states have been firstly proposed and implemented for quantum metrology [3, 13, 17], it is known that a class of two-mode CV states can be decomposed in the superposition of NOON states with different photon numbers. Two interesting examples of these states, called entangled-coherent states (ECSs) [6, 18–25] and entangled states with a squeezed vacuum and a coherent state (EVCs) [29–32], have been recently demonstrated with the current optical technology [25, 32].

The phase uncertainty is limited by the bound of classical Fisher information and allowing quantum measurements provides us the quantity of quantum Fisher information indicating maximised classical Fisher information and minimised the phase uncertainty [37]. Quantum Fisher information implies the optimal positive-operator valued measure (POVM) and gives an ultimate/theoretical lower bound for measuring phase uncertainty in the prepared state. To construct optimal POVM is in general challenging and to consider specific measurement setups could be useful to demonstrate the phase enhancement beyond the classical limit of optimal phase estimation. For example, the parity measurement in pure ECSs can outperform the quantum phase enhancement compared with the limits given by NOON states but does not achieve the saturation of the quantum Fisher information [22].

We present a generalised form of optimal phase estimation using QFI in two-mode CV states with linear/nonlinear phase shifts. We focus on the investigation of the phase sensitivity in both ECSs and EVCs, which have been recently built with the current optical technology. The paper is organised as follows. Section II introduces a background in the optical setup with phase shifts and the aspect of quantum Fisher information. A generalised two-mode nonclassical CV state is presented in Section III and two well-known nonclassical states have been examined and compared for optimal phase estimation in Section IV. In Section V, potential application of ECSs has been investigated for calibrating the strength of nonlinearity. Finally, a summary and remarks are given in Section VI.

II. BACKGROUND

Three physical stages are in general required in quantum metrology as shown in Fig. 1. We first initialise a two-mode quantum state $|\Psi\rangle_{12}$ in the preparation stage and operate a generalized non-linear phase shift $U(\phi, k)$,

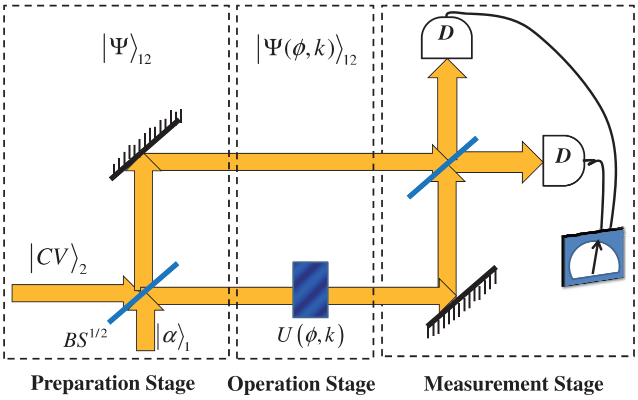

FIG. 1. Schematic of the optical setup

which is applied to mode 2 in the second stage, given by

$$U(\phi, k) = \mathrm{e}^{i\phi(a_2^\dagger a_2)^k}, \tag{1}$$

where $a_i^\dagger$ (a_i) is a creation (annihilation) operator in spatial mode i [23, 33, 34]. The exponent k represents the order of the non-linearity in the phase operation. $k = 1$ corresponds to a linear phase shift on the state while $k \neq 1$ gives a general non-linear effect during the phase operation. One of the well-known examples for a nonlinear phase operation can be achieved through the Kerr-type interaction for $k = 2$ [6, 35]. Several theoretical works have recently been investigated that nonlinearity could help the improvement of phase estimation limits in linear systems [6–11]. Finally, any optimal measurement can be considered at the measurement stage.

With optical interferometric scenarios, we consider the case of applying the phase operation to just one mode, although it should be noted that other cases can be envisaged (see [36] and a footnote in [23]). When the generalized phase operation $U(\phi, k)$ is applied to mode 2 of $|\Psi\rangle_{12}$, the resultant state is equal to

$$|\Psi(\phi, k)\rangle_{12} = (\mathbb{1} \otimes U(\phi, k))|\Psi\rangle_{12}. \tag{2}$$

From phase estimation theory [37], the phase uncertainty is bounded by the quantum Fisher information

$$\delta\phi \geq \frac{1}{\sqrt{F}} \geq \frac{1}{\sqrt{F^Q}}, \tag{3}$$

where F (F^Q) denotes classical (quantum) Fisher information and the value of F^Q (for pure states) is simply given by

$$F^Q = 4\Big[\langle\tilde{\psi}^k|\tilde{\psi}^k\rangle - |\langle\tilde{\psi}^k|\psi^k(\phi)\rangle|^2\Big] = 4(\Delta n^k)^2, \tag{4}$$

with $|\tilde{\psi}^k\rangle = \partial|\psi^k(\phi)\rangle/\partial\phi$ and $(\Delta n^k)^2 = \langle(n^k)^2\rangle - \langle n^k\rangle^2$ ($\langle n^k\rangle = {}_{12}\langle\psi|(a_2^\dagger a_2)^k|\psi\rangle_{12}$). It is important to note two key issues here. 1) For specific measurement scenarios, the number of measurements plays an important role to reach optimal phase estimation [38–40] but one can always find a set of POVM which provides a saturation lower bound with $F = F^Q$ in Eq. (3). 2) $\langle n^1\rangle$ denotes an average (or mean) photon number and the issues of the number of measurements have been addressed in Ref. [22].

III. NONCLASSICAL CONTINUOUS-VARIABLE STATE IN TWO MODES

We here investigate a generalised nonclassical CV state in two modes mixed from two input CV states through a BS in the preparation stage (see Fig. 1) and this scheme has been already demonstrated in optical regime [32]. The advantage of this CV setup is that we are in principle able to tune the parameters/amplitudes of the input CV states continuously, which could provides useful applications for CV quantum metrology in the future (see details in Section V).

We assume that a coherent state and a nonclassical CV state are prepared in each mode such as

$$|\alpha\rangle_1 = \mathrm{e}^{-\frac{|\alpha|^2}{2}} \sum_{n=0}^{\infty} \frac{\alpha^n}{\sqrt{n!}}|n\rangle_1 = \mathrm{e}^{-\frac{|\alpha|^2}{2}} \sum_{n=0}^{\infty} \frac{\alpha^n}{n!}(a_1^\dagger)^n|0\rangle_1, \tag{5}$$

$$|CV\rangle_2 = \sum_{m=0}^{\infty} C_{2m}|2m\rangle_2 = \sum_{m=0}^{\infty} \frac{C_{2m}}{\sqrt{(2m)!}}(a_2^\dagger)^{2m}|0\rangle_2. \tag{6}$$

Note that the nonclassical CV state in mode 2 only contains even photon-number states [29]. After a typical 50:50 BS is applied between the states, the resultant state is equal to

$$\begin{aligned} |\Psi\rangle_{12} &= BS_{1,2}^{1/2}|\alpha\rangle_1|CV\rangle_2, \qquad (7)\\ &= \mathrm{e}^{-\frac{|\alpha|^2}{2}} BS_{1,2}^{1/2} \sum_{n,m=0}^{\infty} \frac{\alpha^n}{n!}\frac{C_{2m}}{\sqrt{(2m)!}}(a_1^\dagger)^n(a_2^\dagger)^{2m}|0\rangle_1|0\rangle_2,\\ &= \mathrm{e}^{-\frac{|\alpha|^2}{2}} \sum_{p=0}^{\infty}\frac{1}{(\sqrt{2})^p}\sum_{q=0}^{p}\Big[\mathcal{R}^\alpha_{p,q}(b_1^\dagger)^q\,(b_2^\dagger)^{p-q}\Big]\,|0\rangle_1|0\rangle_2, \end{aligned}$$

where

$$\mathcal{R}^\alpha_{p,q} = \sum_{m=0}^{[p/2]} \frac{C_{2m}\,\alpha^{p-2m}}{(p-2m)!\sqrt{(2m)!}} \sum_{k'=max[0,q-p+2m]}^{min[2m,q]} (-1)^{k'} \frac{(p-2m)!}{(q-k')!\,(p-q-2m+k')!}\,\frac{(2m)!}{(k')!\,(2m-k')!}. \tag{8}$$

Through the linear/nonlinear phase operation in mode 2, the outcome state is given by

$$|\Psi(\phi,k)\rangle_{12} = e^{-\frac{\alpha^2}{2}} \sum_{p=0}^{\infty} \frac{1}{(\sqrt{2})^p} \sum_{q=0}^{p} \left[\mathrm{e}^{i\phi(p-q)^k} \mathcal{R}^{\alpha}_{p,q} \sqrt{q!}\sqrt{(p-q)!} \right] |q\rangle_1 |p-q\rangle_2. \tag{9}$$

Then, the phase uncertainty of the state is bounded by $1/\sqrt{F^Q_k}$ given by quantum Fisher information

$$\begin{aligned} F^Q_k &= 4\left(\langle\tilde{\Psi}^k|\tilde{\Psi}^k\rangle - |\langle\tilde{\Psi}^k|\Psi(\phi,k)\rangle|^2\right), \\ &= 4\,e^{-\alpha^2}\left[\sum_{p=0}^{\infty}\frac{1}{2^p}\sum_{q=0}^{p}(\mathcal{R}^{\alpha}_{p,q})^2 q!(p-q)!(p-q)^{2k} - e^{-\alpha^2}\left|\sum_{p=0}^{\infty}\frac{1}{2^p}\sum_{q=0}^{p}(\mathcal{R}^{\alpha}_{p,q})^2 q!(p-q)!(p-q)^{k}\right|^2\right], \end{aligned} \tag{10}$$

where

$$|\tilde{\Psi}^k\rangle = \partial|\Psi(\phi,k)\rangle/\partial\phi = e^{-\frac{\alpha^2}{2}} \sum_{p=0}^{\infty} \frac{i(p-q)^k}{(\sqrt{2})^p} \sum_{q=0}^{p} \left[(p-q)^k \mathrm{e}^{i\phi(p-q)^k} \mathcal{R}^{\alpha}_{p,q} \sqrt{q!}\sqrt{(p-q)!} \right] |q\rangle_1 |p-q\rangle_2. \tag{11}$$

IV. TWO EXAMPLES

A. Case 1 : Coherent state superposition (CSS)

One of the interesting input states in mode 2 is a coherent state superposition (CSS) [5] (also called Schrödinger cat state) given by

$$|CSS_{\pm}(\alpha')\rangle = N^{\pm}_{\alpha'}(|\alpha'\rangle \pm |-\alpha'\rangle), \tag{12}$$

where $|\alpha'\rangle$ is a coherent state with amplitude α' and

$$N^{\pm}_{\alpha'} = 1/\sqrt{2(1\pm \mathrm{e}^{-2|\alpha'|^2})}. \tag{13}$$

Even/odd CSSs with small α' have been already demonstrated in experiments [25, 41] and we particularly focus on the even CSS because it consists of even photon numbers only as given in Eq. (6).

In order to calculate the coefficient $\mathcal{R}^{\alpha'}_{p,q}$ in Eq. (8), $|CSS_+(\alpha')\rangle_2$ is rewritten by

$$|CSS_+(\alpha')\rangle_2 = \sum_{m=0}^{\infty} C^{css}_{2m}|2m\rangle_2. \tag{14}$$

and

$$C^{css}_{2m} = e^{-\frac{\alpha'^2}{2}} \frac{2N^{+}_{\alpha'}(\alpha')^{2m}}{\sqrt{(2m)!}}. \tag{15}$$

Then, when the coherent state $|\alpha\rangle$ is fed with the even CSS through a BS, the initialised state at the preparation stage is given by

$$|\Psi_{ECS}(\alpha,\alpha')\rangle = BS^{1/2}_{1,2}|\alpha\rangle_1|CSS_+(\alpha')\rangle_2. \tag{16}$$

It is well-known that a NOON-type typical ECS can be generated by mixing a coherent state with a CSS through a BS. For $\alpha' = \alpha$, the resultant two-mode states from $|CSS_+(\alpha)\rangle_2$ and $|\alpha\rangle_1$ become the typical even ECSs given by

$$|\Psi_{ECS}(\alpha,\alpha)\rangle_{12} = N^{+}_{\alpha}\left[|\sqrt{2}\alpha\rangle_1|0\rangle_2 + |0\rangle_1|\sqrt{2}\alpha\rangle_2\right]. \tag{17}$$

After a phase shifter, it can be represented by

$$\begin{aligned} |\Psi_{ECS}(\alpha,\alpha,\phi,k)\rangle_{12} &= [\mathbb{1}\otimes U(\phi,k)]\,|\Psi_{ECS}(\alpha,\alpha)\rangle_{12}, \\ &= \mathrm{e}^{-|\alpha|^2} N^{+}_{\alpha} \sum_{n=0}^{\infty} \frac{(\sqrt{2}\alpha)^n}{n!} \left[(a^{\dagger}_1)^n + \mathrm{e}^{i\phi(n)^k}(a^{\dagger}_2)^n\right] |0\rangle_1|0\rangle_2. \end{aligned} \tag{18}$$

B. Case 2 : Squeezed vacuum states (SVSs)

Instead of $|CSS_+(\alpha')\rangle_2$ above, a squeezed vacuum state (SVS) is an alternative nonclassical state given by

$$|SVS(r)\rangle_2 = \sum_{m=0}^{\infty} C^{svs}_{2m}|2m\rangle_2, \tag{19}$$

where

$$C^{svs}_{2m} = \sqrt{\frac{(2m)!}{\cosh r}}\frac{(-1)^m}{2^m m!}(\tanh r)^m. \tag{20}$$

Several experiments show that squeezing operation through second-harmonic generation can be achieved with the current technology [26–28]. Applying a BS, the entangled state from a squeezed vacuum and a coherent state (EVC) is given by

$$|\Psi_{EVC}(\alpha,r)\rangle_{12} = BS^{1/2}_{1,2}|\alpha\rangle_1|SVS(r)\rangle_2, \tag{21}$$

through a phase shifter in mode 2, it becomes

$$|\Psi_{EVC}(\alpha,r,\phi,k))\rangle_{12} = [\mathbb{1}\otimes U(\phi,k)]\,|\Psi_{EVC}(\alpha,r)\rangle_{12}. \tag{22}$$

Recently, the two-mode EVC has been theoretically investigated for linear phase shift with/without particle losses [29] and experimentally examined in order to demonstrate the characteristics of multi-photon states

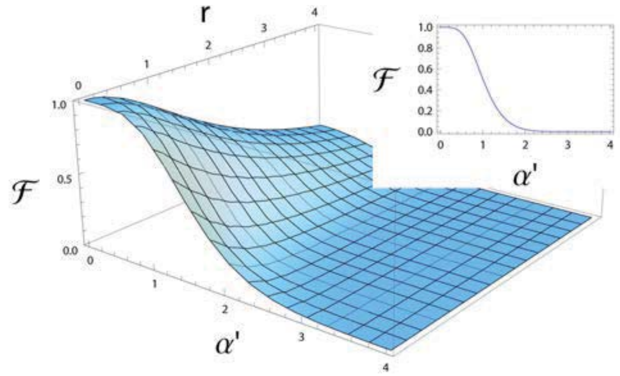

FIG. 2. Fidelity of CSS and SVS ($\mathcal{F}(\alpha', r)$) with the inset figure given in Eq. (27). From the inset, the fidelity is close to 1 for $\alpha' < 0.5$ when the average photon numbers of CSS and SVS are the same.

[32].

C. Comparison of QFI between ECS and EVC

To compare the phase sensitivity among the different quantum states, the same average photon number can be used as the physical resource count for the states [6, 22, 24]. For ECS and EVC, to match the photon numbers of CSS and SVS is required for because the other input in mode 1 is a coherent state $|\alpha\rangle_1$ as a common ingredient in the setup. The relationship between α' and r for the same average photon number of $|CSS_+(\alpha')\rangle_2$ and $|SVS(r)\rangle_2$ is following.

$$\sum_{m=0}^{\infty} 2m\,|C_{2m}^{css}|^2 = \sum_{m=0}^{\infty} 2m\,|C_{2m}^{svs}|^2 \tag{23}$$

$$\alpha'^2 \tanh \alpha'^2 = \sinh^2 r. \tag{24}$$

Thus,

$$r = \operatorname{arcsin\,h}\left(\sqrt{\alpha'^2 \tanh \alpha'^2}\right). \tag{25}$$

As shown in Fig. 2, the fidelity is given by

$$\begin{aligned}\mathcal{F}(\alpha', r) &= \langle CSS_+(\alpha')|SVS(r)\rangle \\ &= \frac{2N_{\alpha'}^{+}}{\sqrt{\cosh r}} \exp\left[-\frac{\alpha'^2}{2}(\tanh r + 1)\right]. \end{aligned}\tag{26}$$

Note that the states with small α' and r approach to a vacuum state and $\mathcal{F} \approx 1$. If the average photon numbers are matched as given in Eq. (25), the fidelity becomes

$$\mathcal{F}(\alpha') = \frac{2N_{\alpha'}^{+} \exp\left[-\alpha'^2/2\right]}{(1 + \alpha'^2 \tanh \alpha'^2)^{1/4}} \exp\left[-\frac{\alpha'^3}{2\sqrt{\alpha'^2 + \coth \alpha'^2}}\right], \tag{27}$$

drawn in the inset of Fig. 2.

The quantum Fisher information of ECSs given by

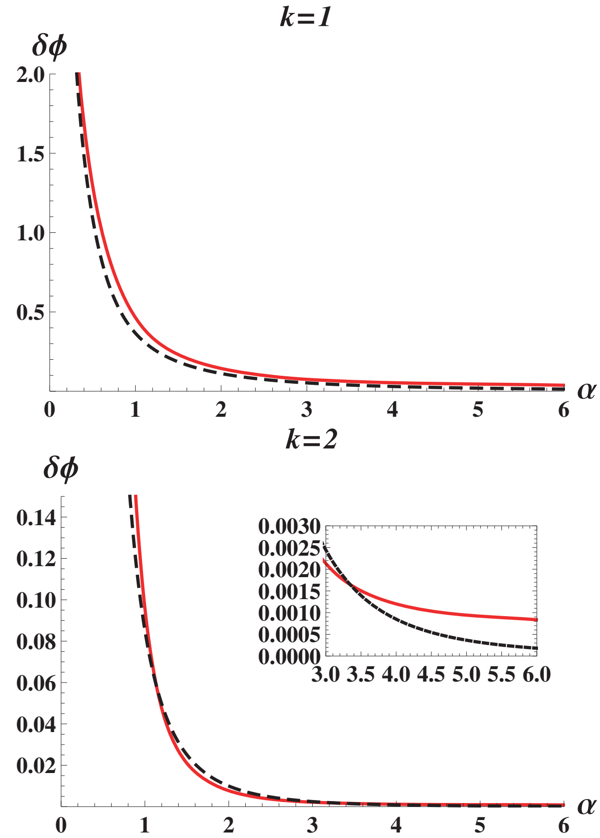

FIG. 3. The bound of phase uncertainty $1/\sqrt{F^Q}$ with respect to α for $k = 1, 2$ (the solid line donotes EVC and the dashed one does ECS).

Eq. (10) is equal to

$$F_{ECS}^{Q}(k) = 4\big[\langle E'|E'\rangle - |\langle E'|\Psi_{ECS}(\alpha, \alpha, \phi, k)\rangle|^2\big], \tag{28}$$

for $|E'\rangle = \partial|\Psi_{ECS}(\alpha, \alpha, \phi, k)\rangle/\partial\phi$ and $\alpha = \alpha'$ for simplicity. For EVCs, we calculate $\mathcal{R}_{p,q}^{\alpha}$ and the quantum Fisher information is given by

$$F_{EVC}^{Q}(k) = 4\big[\langle V'|V'\rangle - |\langle V'|\Psi_{EVC}(\alpha, r_0, \phi, k)\rangle|^2\big], \tag{29}$$

where $|V'\rangle = \partial|\Psi_{EVC}(\alpha, r, \phi, k)\rangle/\partial\phi$ and $r_0 = \operatorname{arcsin\,h}\left(\sqrt{\alpha^2 \tanh \alpha^2}\right)$ for the same average photon number.

In Fig. 3, we numerically obtain the lower bound of phase uncertainty of ECS and EVC given by QFI for $k = 1, 2$. For $\alpha < 0.5$, the cureves of ECS approach to that of EVC due to their high fidelity. For $k = 1$, the phase uncertainty of ECS is always better that of EVC for all α while some interesting crossing points exist for $k = 2$ (see the inset of Fig. 3).

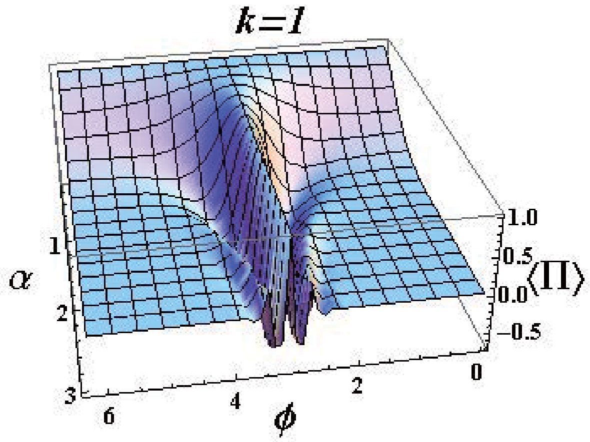

FIG. 4. $\langle\hat{\Pi}_{ECS}(k)\rangle$ for $k=1$

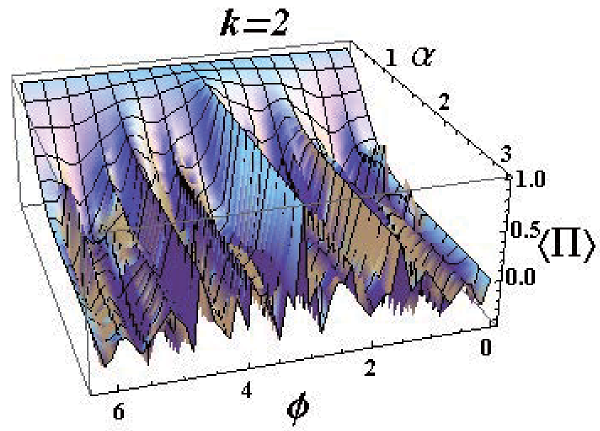

FIG. 5. $\langle\hat{\Pi}_{ECS}(k)\rangle$ for $k=2$

V. APPLICATIONS: QUANTUM-ENHANCED CALIBRATION OF NON-LINEARITIES

Toward practical applications of ECSs, one can calibrate the strength of un-calibrated nonlinearity in a medium by measuring phase $\phi^{(k)}$ for nonlinearity k [23]. In optimal phase estimation, known/calibrated nonlinear medium can be utilized for the enhancement of linear phase estimation [6, 8–11] and also has potential single-photon detection without absorption [42–44] and optical quantum computation [45]. Thus, the method of nonlinearity calibration could give significant motivation for the application of the nonclassical CV states in quantum metrology.

Since the parity measurement using EVCs with a linear phase shift has been recently studied [31], we here focus on the case of applying parity measurements using ECSs for $k=1,2$. The parity measurement in mode 2 is given by the expectation value

$$\langle\hat{\Pi}(k)\rangle = {}_{12}\langle\Psi(\phi,k)|(BS^{1/2}_{1,2})^{\dagger}\,\hat{\Pi}\,BS^{1/2}_{1,2}|\Psi(\phi,k)\rangle_{12}, \quad (30)$$

where $\hat{\Pi}=\mathrm{e}^{i\pi a_2^{\dagger}a_2}$. For ECSs in Eq. (18), the expectation value is given by

$$\langle\hat{\Pi}_{ECS}(k)\rangle = \mathcal{M}_{\alpha}\left(4+\sum_{n=1}^{\infty}\frac{2(-1)^n}{n!}\left(2\alpha^2\right)^n\cos\left(n^k\phi\right)\right), \quad (31)$$

where $\mathcal{M}_{\alpha}=\left(\mathrm{e}^{-|\alpha|^2}N_{\alpha}^{+}\right)^2$ (e.g., the case of $k=1$ is given in [22]). We here take the advantage of CV states that we can continuously measure the expectation values of parity measurement by changing the parameters of the states (α and ϕ). For example, the parameter ϕ and the amplitude α can be shifted by the length of the medium and the strength of the input coherent state.

First we measure the amount of linear phase shift ($k=1$) in a thin unknown medium using a calibrated linear material (e.g., crystal) with $\phi_0^{(1)}=\pi$ linear phase shift (we here assume that the calibrated medium dominantly produces a linear phase shift and nonlinear phase shift is ignorable in the medium). Once a ECS is prepared with small α, which could be able to minimize the effect of nonlinear phase shift in the unknown medium, the thin unknown and the calibrated known materials are both located in mode 2. If we tune the amplitude α and with different $\phi^{(1)}$, the results of the parity measurement give a curve similar to the central feature at $\phi^{(1)}+\phi_0^{(1)}\approx\pi$ in Fig. 4. Because the curvature rapidly changes at around $\phi=\pi$ and provides approximated minimum phase uncertainty for $k=1$ [22], one expect to obtain the vaule of $\phi^{(1)}$ very accurately.

Since the amount of the linear phase shift $\phi^{(1)}$ is now known precisely, we can compensate the linear phase shift given by the unknown medium using another calibrated linear shifter with $-\phi^{(1)}$ in order to emphasise the effect of the unknown nonlinear phase shift. Therefore, after inserting additional the calibrated linear phase shifter for compensation of the linear phase shift in the unknown medium, we re-perform the parity measurement for $k=2$ to obtain the expectation value, which can be compared with the results in Fig. 5.

In addition, we can double-check the validity of the results using the phase variance of parity measurement given by

$$\Delta\phi(k) = \frac{1-\langle\hat{\Pi}(k)\rangle^2}{|\partial\langle\hat{\Pi}(k)\rangle/\partial\phi|^2}. \quad (32)$$

As shown in Fig. 6, or ECSs, the phase variance by parity measurement is given by

$$\Delta\phi_{ECS}(k) = \frac{1-\langle\hat{\Pi}_{ECS}(k)\rangle^2}{|\partial\langle\hat{\Pi}_{ECS}(k)\rangle/\partial\phi|^2}. \quad (33)$$

The curves in Fig. 6 show that the phase uncertainty given by parity measurements for large α approaches to the bound of quantum Fisher information.

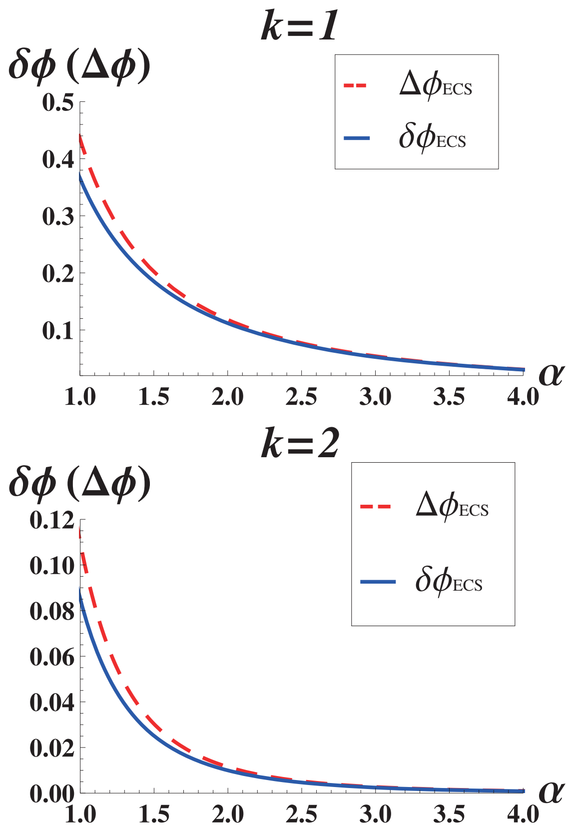

FIG. 6. $\Delta\phi_{ECS}(k)$ compared with the bound of $\delta\phi_{ECS}(k)$ with $\phi^{(k)} = \pi - 0.001$ for $k = 1, 2$

VI. SUMMARY AND REMARKS

In summary, we have investigated the enhanced phase estimation in generalised nonclaisscal CV states in linear/nonlinear phase operation. Two non-trivial examples (such as $|ECS\rangle$ and $|EVC\rangle$) are rigorously studied for phase uncertainty in terms of quantum Fisher information and the idea of potential application for calibration of nonlinearity has been investigated for quantum metrology.

In practice, particle losses may result in imperfect linear/nonlinear phase operations in the setup, especially for $k \neq 1$, however, the robustness of ECSs against particle losses has been however known for $k = 1, 2$ [23, 30, 46–49]. Note that the saturation of the phase variance given by parity measurement only occurs for $\phi^{(2)} \approx \pi$ for $k = 2$ in Fig. 6 and the amount of nonlinear phase shift is still very challenging to produce with the current optical technology.

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